Superrigidity of group von Neumann algebras

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Abstract

Over the last years, Popa's deformation/rigidity theory led to a lot of progress in the classification of *group measure space II_1 factors* $\mathrm{L}^\infty(X) \rtimes \Gamma$ associated with free, ergodic, probability measure preserving actions of countable groups. In comparison, our understanding of *group von Neumann algebras* $\mathrm{L}\Gamma$ is much more limited.

One of the fundamental problems in the theory of II_1 factors is to classify the group von Neumann algebras $\mathrm{L}\Gamma$ in terms of the group Γ. More precisely, we want to know how much $\mathrm{L}\Gamma$ remembers about the group Γ. A celebrated theorem of Connes from 1976 says that whenever Γ is i.c.c. amenable, the group factor $\mathrm{L}\Gamma$ does not remember anything about the group, except its amenability. The opposite phenomenon, when $\mathrm{L}\Gamma$ remembers everything about Γ, is called *W^*-superrigidity*. Connes' rigidity conjecture from 1980 says that i.c.c. groups with Kazhdan's property (T) are W*-superrigid, but this remains wide open even for classical groups like $\mathrm{SL}_n(\mathbb{Z})$, with $n \geq 3$.

A countable group $\mathcal{G}$ is called W*-superrigid if for any countable group Λ such that $\mathrm{L}\mathcal{G} \cong \mathrm{L}\Lambda$ we have that the groups $\mathcal{G}$ and Λ are isomorphic. The first example of such W*-superrigid groups was given only in 2010 by Ioana, Popa and Vaes. They proved that for a large class of *generalized wreath products* $\mathcal{G}$, the group factor $\mathrm{L}\mathcal{G}$ completely remembers the group $\mathcal{G}$. The class of groups covered by their work contains all wreath products $\mathcal{G} = (\mathbb{Z}/2\mathbb{Z})^{(I)} \rtimes (\Gamma \wr \mathbb{Z})$, where Γ is an arbitrary non-amenable group and $I = (\Gamma \wr \mathbb{Z})/\mathbb{Z}$.

Motivated by the work of Ioana, Popa and Vaes, we find in this thesis more natural examples of W*-superrigid groups. Given a countable group Γ, we consider the action of the direct product $\Gamma \times \Gamma$ on Γ by left-right multiplication and we define the generalized wreath product group $\mathcal{G} := H^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$, where $H = \mathbb{Z}/2\mathbb{Z}$. We prove that $\mathcal{G}$ is W*-superrigid whenever Γ belongs to a large class of non-amenable groups, containing free groups, hyperbolic groups, non-trivial free products, certain groups with positive first ℓ^2-Betti number, etc.

Contents

Chapter 1

Introduction and main results

1.1 II_1 factors and their classification

A von Neumann algebra is an algebra of bounded operators on a Hilbert space that is closed under taking adjoint elements and that is closed in the strong operator topology. Von Neumann algebras arise naturally in the study of groups and their actions on probability measure spaces. If Γ is a countable group, then the image $(u_g)_{g\in\Gamma}$ of the left regular representation of Γ on the Hilbert space $\ell^2(\Gamma)$ generates the *group von Neumann algebra* $\mathrm{L}\Gamma$. If Γ is a countable group acting on a probability measure space (X,μ) by probability measure preserving transformations, then it gives rise to the *group measure space von Neumann algebra* $\mathrm{L}^\infty(X)\rtimes\Gamma$, which is generated by $\mathrm{L}^\infty(X)$ and unitary elements $(u_g)_{g\in\Gamma}$ satisfying $u_g^* f(\cdot)u_g = f(g\cdot)$ and $u_g u_h = u_{gh}$, for all $g,h\in\Gamma$ and $f\in\mathrm{L}^\infty(X)$. These constructions go back to Murray and von Neumann's seminal papers [MvN36] and [MvN43] and provide a very rich source of examples.

One of the main problems in the theory of von Neumann algebras is to classify $\mathrm{L}\Gamma$ and $\mathrm{L}^\infty(X)\rtimes\Gamma$ in terms of the group Γ, respectively in terms of the group action $\Gamma \curvearrowright (X,\mu)$. In this thesis we will focus more on group von Neumann algebras, but the classification of group measure space von Neumann algebras will also come into play.

A *factor* is a von Neumann algebra with trivial center. Factors are exactly the "simple" von Neumann algebras, namely those von Neumann algebras that cannot be written as a direct sum of two. Murray and von Neumann have classified factors into three types [MvN36] and have proven that every von Neumann algebra can be decomposed as a direct integral of factors [vN49].

Connes [Co72] proved that general factors can be built up from a special category of factors, called *II_1 factors*. A II_1 factor is an infinite dimensional factor that admits a finite positive *trace*. For instance, the group von Neumann algebra $\mathrm{L}\Gamma$ is a II_1 factor if and only if Γ is infinite and all of its conjugacy classes, except for the trivial one, are infinite (shortly, we say that Γ is i.c.c.). The group measure space construction $\mathrm{L}^\infty(X) \rtimes \Gamma$ is a II_1 factor if the action $\Gamma \curvearrowright (X, \mu)$ is *essentially free* and *ergodic*. The action $\Gamma \curvearrowright (X, \mu)$ is essentially free if the set of fixed points of every non-trivial element in Γ is negligible and ergodic if any globally Γ-invariant measurable subset of X is either negligible or co-negligible.

Murray and von Neumann [MvN43] proved the existence of a unique *hyperfinite* II_1 factor $\mathcal{R}$, up to isomorphism, defined as the direct limit of an increasing sequence of matrix algebras, or equivalently, as the group von Neumann algebra of the infinite symmetric group $S(\infty)$. Connes' famous result [Co76] implies that there is a unique *amenable* II_1 factor, up to isomorphism. In other words, the hyperfinite factors are isolated among all II_1 factors by their amenability. Since $\mathrm{L}\Gamma$ is amenable if and only if the group Γ is amenable, it follows that all the group von Neumann algebras arising from i.c.c. amenable groups are isomorphic.

Distinguishing between group von Neumann algebras, or between II_1 factors, in general, is very difficult. In the early years, examples of different groups factors $\mathrm{L}\Gamma$ were obtained by studying *central sequences* of elements in $\mathrm{L}\Gamma$. Using an invariant called *property Gamma* that is defined in terms of central sequences, Murray and von Neumann [MvN43] proved that the free group factors $\mathrm{L}\mathbb{F}_n$ are not hyperfinite. Very few non-isomorphic II_1 factors were discovered until 1969 when McDuff [McD69] constructed an uncountable family of non-isomorphic II_1 factors. It is still an open problem whether the free group factors $\mathrm{L}\mathbb{F}_n$ and $\mathrm{L}\mathbb{F}_m$ are isomorphic or not for different n and m. Nevertheless, the work of Dykema [Dy94] and Rădulescu [Ra84] implies that either they are all pairwise isomorphic or all pairwise non-isomorphic.

Over the years, surprising isomorphisms between group von Neumann algebras have appeared. As we already mentioned, Connes [Co76] proved that all group II_1 factors arising from i.c.c. amenable groups are isomorphic to the hyperfinite II_1 factor. This shows that amenable groups manifest a remarkable *softness*: all algebraic properties of the groups, except amenability, are lost when passing to the group von Neumann algebra. Generally speaking, a group Γ is *soft* when its group von Neumann algebra $\mathrm{L}\Gamma$ does not "remember" the group.

Voiculescu's free probability theory led to other surprising isomorphisms of group factors. Dykema [Dy93] proved that for $n \geq 2$ and $\Gamma_1, \ldots, \Gamma_n$ infinite amenable groups, the group von Neumann algebra of their free product $\mathrm{L}(\Gamma_1 * \ldots * \Gamma_n)$

is isomorphic to the free group factor $\mathrm{L}\mathbb{F}_n$. Dykema and Rădulescu showed in [DR00] that if $\Gamma = \Gamma_1 * \Gamma_2 * \ldots$ is an infinite free product of non-trivial groups Γ_i, then $\mathrm{L}\Gamma$ is isomorphic to $\mathrm{L}(\mathbb{F}_\infty * \Gamma)$.

Ioana [Io06] proved that the group von Neumann algebras $\mathrm{L}(\mathbb{F}_n \wr \mathbb{Z})$, associated to *wreath products* $\mathbb{F}_n \wr \mathbb{Z}$, are all isomorphic, for $n \geq 2$. The wreath product $H \wr \Gamma$ of two countable groups H and Γ is defined as the semidirect product $H^{(\Gamma)} \rtimes \Gamma$, where $H^{(\Gamma)}$ denotes the direct sum of copies of H indexed by Γ and where Γ acts on $H^{(\Gamma)}$ by shifting the indices. Wreath products are closely related to *Bernoulli actions*. If Γ is a countable group and (X_0, μ_0) is a probability space then the action of Γ on the product space $(X_0, \mu_0)^\Gamma$ by shifting the indices is called the Bernoulli action. If H is abelian, then its group von Neumann algebra $\mathrm{L}H$ can be identified with $\mathrm{L}^\infty(\widehat{H})$, where $\widehat{H}$ is the Pontryagin dual of H, equipped with the Haar probability measure, and therefore we have the identification $\mathrm{L}(H \wr \Gamma) \cong \mathrm{L}^\infty(\widehat{H}^\Gamma) \rtimes \Gamma$. Bowen [Bo09a], [Bo09b] showed that, for fixed $n \geq 1$, any two Bernoulli actions $\mathbb{F}_n \curvearrowright (X_0, \mu_0)^{\mathbb{F}_n}$ and $\mathbb{F}_n \curvearrowright (Y_0, \nu_0)^{\mathbb{F}_n}$ of the same free group $\mathbb{F}_n$, but with different base spaces, are *orbit equivalent*, i.e. there is an isomorphism $\theta : (X_0, \mu_0) \to (Y_0, \nu_0)$ that sends orbits to orbits. As a consequence, we have that all $\mathrm{L}(H \wr \mathbb{F}_n)$, for H non-trivial abelian, are isomorphic.

1.2 Popa's deformation/rigidity theory

Popa's deformation/rigidity theory initiated in [Po01] was a major breakthrough in the classification of II_1 factors. The fundamental idea of this theory is to study von Neumann algebras that have, at the same time, *rigid* parts and strong *deformation* properties. The first major result obtained by Popa using these methods was the discovery of the first example of a II_1 factor with trivial *fundamental group* in [Po01]. In [Po03] and [Po04] he made striking progress towards the classification of II_1 factors by proving *strong rigidity* theorems for group measure space von Neumann algebras associated to Bernoulli actions of *property (T)* groups, as we shall explain below.

A *deformation* of a von Neumann algebra is a sequence of completely positive maps that converges to the identity pointwise σ-weakly. Both the group von Neumann algebra $\mathrm{L}\Gamma$ and the group measure space construction $\mathrm{L}^\infty(X) \rtimes \Gamma$ admit natural deformations given by positive definite functions on Γ. If $f : \Gamma \to \mathbb{C}$ is a positive definite function, then the maps $u_g \mapsto f(g) u_g$ and $a u_g \mapsto f(g) a u_g$ extend to completely positive maps on $\mathrm{L}\Gamma$, respectively on $\mathrm{L}^\infty(X) \rtimes \Gamma$. For instance, if Γ has the *Haagerup approximation property*, there exits a sequence $f_n : \Gamma \to \mathbb{C}$ of positive definite functions on Γ such that $f_n \in c_0(\Gamma)$ and such that f_n converges to 1 pointwise. This kind of deformations played a crucial role in [Po01].

Another source of deformations is given by Popa's *malleable deformations* for Bernoulli actions. Such deformations were the main ingredient in proving strong rigidity results for group measure space II_1 factors in [Po03], [Po04]. A variant of Popa's malleable deformation is the *tensor length deformation* for *generalized Bernoulli actions* considered by Ioana [Io06]. If $\Gamma \curvearrowright I$ is an action of a countable group Γ on a countable set I and (X_0, μ_0) is a base probability space, then the generalized Bernoulli action $\Gamma \curvearrowright (X, \mu) := (X_0, \mu_0)^I$ is defined by shifting the indices. If we denote $A_0 := \mathrm{L}^\infty(X_0)$ and $A := \mathrm{L}^\infty(X) = A_0^I$, then, for $0 < \rho < 1$, one can define a deformation θ_ρ on A by $\theta_\rho(a) = \rho^n a$, whenever $a \in (A_0 \ominus \mathbb{C}1)^J$, for $J \subset I$ such that $|J| = n$. Then θ_ρ commutes with the generalized Bernoulli action $\Gamma \curvearrowright A_0^I$ and defines a deformation of the group measure space construction $A \rtimes \Gamma$ by $au_g \mapsto \theta_\rho(a)u_g$, for all $a \in A$ and $g \in \Gamma$. These maps also satisfy a strong deformation property, called *malleability*. We will study in greater details this type of deformations in Chapter 3. In [Si10], a similar malleable deformation was defined for a *Gaussian action* associated to an orthogonal representation of Γ, see Chapter 4.

The main source of rigidity used in [Po03] and [Po04] was the *relative property (T)* for inclusions of tracial von Neumann algebras. If M is a von Neumann algebra that admits a finite trace τ and Q is a von Neumann subalgebra of M, then the inclusion $Q \subset M$ has relative property (T) or is *rigid* if any deformation on M converges uniformly to the identity on the unit ball of Q, in the L^2-norm corresponding to the trace τ. Another source of rigidity is *Popa's spectral gap rigidity* for Bernoulli actions from [Po06b]. Recall that a unitary representation of a countable group is said to have *spectral gap* if it does not have non-zero almost invariant vectors. Popa defined a similar notion of spectral gap for inclusions of tracial von Neumann algebras and used it to prove strong rigidity results for Bernoulli actions of direct products of non-amenable groups in [Po06b]. These methods were extended later to generalized Bernoulli actions [PV06], [Va07], [Io10], [IPV10].

As we mentioned before, the fundamental idea of Popa's deformation/rigidity theory is to study von Neumann algebras having both rigid parts and deformation properties. The tension between deformation and rigidity allows us to precisely locate the rigid parts within the ambient von Neumann algebra. In [Po03], [Po04] Popa proved that whenever $\Gamma \curvearrowright (X, \mu)$ is a Bernoulli action of an i.c.c. group Γ and $\Lambda \curvearrowright (Y, \nu)$ is a probability measure preserving (p.m.p.) free ergodic action of a property (T) group Λ such that $\mathrm{L}^\infty(X) \rtimes \Gamma \cong \mathrm{L}^\infty(Y) \rtimes \Lambda$, the groups must be isomorphic and the actions must be conjugate, i.e. there exist isomorphisms $\theta : (X, \mu) \to (Y, \nu)$ and $\delta : \Gamma \to \Lambda$ such that $\theta(g \cdot x) = \delta(g) \cdot \theta(x)$, for all $g \in \Gamma$ and a.e. $x \in X$. In this case the deformation is given by the

Bernoulli action of Γ and the rigidity is given by the property (T) of Λ.

Popa [Po04] asked the natural question whether Bernoulli actions of i.c.c. property (T) groups are *W*-superrigid*. More precisely, if $\Gamma \curvearrowright (X, \mu)$ is a Bernoulli action of an i.c.c. property (T) group Γ and $\Lambda \curvearrowright (Y, \nu)$ is an arbitrary p.m.p. free ergodic action such that $\mathrm{L}^\infty(X) \rtimes \Gamma \cong \mathrm{L}^\infty(Y) \rtimes \Lambda$, are then the groups isomorphic and the actions conjugate ? This question has been answered affirmatively by Ioana [Io10]. The strategy of proving W*-superrigidity for Bernoulli actions is particularly important for us, so we will briefly describe it. Let $\Gamma \curvearrowright (X, \mu)$ be a Bernoulli action of an i.c.c. property (T) group Γ. Let $M := \mathrm{L}^\infty(X) \rtimes \Gamma$ and assume that $M \cong \mathrm{L}^\infty(Y) \rtimes \Lambda$ for an arbitrary p.m.p. free ergodic action $\Lambda \curvearrowright (Y, \nu)$. The first step consists of classifying all unital $*$-homomorphisms $M \to M \overline{\otimes} M$ in terms of the decomposition $M = \mathrm{L}^\infty(X) \rtimes \Gamma$ and getting some rigidity concerning their structure. In the second step consider the *comultiplication* type embedding $\Delta : M \to M \overline{\otimes} M$ defined in [PV09] by $\Delta(au_s) = au_s \otimes u_s$, for all $a \in \mathrm{L}^\infty(Y)$ and $s \in \Lambda$. Applying the classification result from the first step, we obtain information about Δ with respect to both group measure space decompositions of M and, ideally, this information will be powerful enough to imply that the two decompositions coincide. These methods have been successfully extended to generalized Bernoulli actions in [IPV10], as we shall see in the next section.

1.3 Superrigidity of group II_1 factors

The first *rigidity* phenomena for von Neumann algebras were discovered by Connes in [Co80a], [Co80b], where he asked whether two i.c.c. property (T) groups Γ and Λ, with isomorphic group von Neumann algebras $\mathrm{L}\Gamma \cong \mathrm{L}\Lambda$, must necessarily be isomorphic. This question is nowadays referred to as *Connes' rigidity conjecture*. Actually, more than distinguishing between property (T) group factors, a positive answer to this question implies that the group factor $\mathrm{L}\Gamma$ of an i.c.c. property (T) group Γ uniquely determines the group Γ. This conjecture remains wide open, even for classical groups like $\mathrm{SL}(n, \mathbb{Z})$, for $n \geq 3$. Remark however that whenever Γ and Λ are lattices in $\mathrm{Sp}(n, 1)$, respectively $\mathrm{Sp}(m, 1)$, the isomorphism of $\mathrm{L}\Gamma$ and $\mathrm{L}\Lambda$ implies that $n = m$, cf. [CH88].

Another class of groups that have shown remarkable rigidity in the context of von Neumann algebras is the class of wreath product groups. Popa [Po04] proved that whenever Γ_i are i.c.c. property (T) groups, for $i = 1, 2$, such that the wreath product groups $G_i = \mathbb{Z}/2\mathbb{Z} \wr \Gamma_i$ have isomorphic group von Neumann algebras, we must have that $G_1 \cong G_2$. If H is a countable group and $\Gamma \curvearrowright I$ is an action of a countable group Γ on a countable set I, then the *generalized* *wreath product* $H \wr_I \Gamma$ is defined as the semidirect product $H^{(I)} \rtimes \Gamma$, where Γ acts on $H^{(I)}$ by shifting the indices. As for plain wreath products, this construction is closely related to generalized Bernoulli actions. More precisely, when H is abelian, the group von Neumann algebra $\mathrm{L}(H \wr_I \Gamma)$ can be written as the group measure space construction $\mathrm{L}^\infty(X^I) \rtimes \Gamma$, where $X = \widehat{H}$ is the Pontryagin dual of H, equipped with the Haar probability measure, and where $\Gamma \curvearrowright X^I$ is the generalized Bernoulli action.

Despite the striking progress concerning W*-superrigidity for group actions ([PV09], [Io10]), the problem for groups is much harder. Only in 2010 Ioana, Popa and Vaes [IPV10] could establish the first *W*-superrigidity* theorem for group von Neumann algebras: for a large class of generalized wreath product groups $G = (\mathbb{Z}/2\mathbb{Z})^{(I)} \rtimes \Gamma$, it was shown that if $\mathrm{L}G \cong \mathrm{L}\Lambda$, for an arbitrary countable group Λ, then G must be isomorphic with Λ. Such a group G is said to be *W*-superrigid* (see Definition 1.1), and in this case the group von Neumann algebra $\mathrm{L}G$ completely remembers G. A particular example of W*-superrigid group in [IPV10] is the generalized wreath product $G = (\mathbb{Z}/2\mathbb{Z})^{(I)} \rtimes (\mathbb{F}_n \wr \mathbb{Z})$, where $I := (\mathbb{F}_n \wr \mathbb{Z})/\mathbb{Z}$, which contrasts with [Io06], where all $\mathrm{L}(\mathbb{F}_n \wr \mathbb{Z})$ were shown to be isomorphic for different $n \geq 2$. Actually one might wonder if such methods could also work for plain wreath products. For example, motivated by [Io10], one might hope to prove W*-superrigidity for wreath products $H \wr \Gamma$ of i.c.c. property (T) groups, but this does not work. In [IPV10] is proven that if Γ is an i.c.c. property (T) group and H is a non-trivial abelian group such that $\mathrm{L}(H \wr \Gamma) \cong \mathrm{L}\Lambda$, for an arbitrary group Λ, then there exists a non-trivial abelian group Σ on which Γ acts by automorphisms and such that $\Lambda \cong \Sigma \rtimes \Gamma$ and the actions $\Gamma \curvearrowright \widehat{\Sigma}$ and $\Gamma \curvearrowright \widehat{H}^{(\Gamma)}$ are conjugate. Typically there are lots of such groups Σ, thus $\mathrm{L}(H \wr \Gamma) \cong \mathrm{L}(\Sigma \rtimes \Gamma)$ for many non-isomorphic groups Σ.

The precise definition of W*-superrigidity for groups goes as follows.

Definition 1.1. A countable group $\mathcal{G}$ is called *W*-superrigid* if the following holds: if Λ is any countable group and if $\pi : \mathrm{L}\Lambda \to (\mathrm{L}\mathcal{G})^r$ is a $*$-isomorphism for some $r > 0$, then $r = 1$ and there exist a group isomorphism $\delta : \Lambda \to \mathcal{G}$, a character $\omega : \Lambda \to \mathbb{T}$ and a unitary $w \in \mathrm{L}\mathcal{G}$ such that

$$\pi(v_s) = \omega(s)\, w\, u_{\delta(s)}\, w^* \quad \text{for all } \ s \in \Lambda\ .$$

Here $(v_s)_{s\in\Lambda}$ and $(u_g)_{g\in\mathcal{G}}$ denote the canonical generating unitaries of $\mathrm{L}\Lambda$, respectively $\mathrm{L}\mathcal{G}$.

Following the same strategy as in [IPV10], we have proven in [BV12] and [Be14] that the more natural *left-right wreath product* groups $\mathcal{G} = (\mathbb{Z}/2\mathbb{Z})^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$ are W*-superrigid, where the direct product $\Gamma \times \Gamma$ acts on Γ by left-right multiplication, and where Γ ranges over a large class of non-amenable countable groups containing the free groups $\mathbb{F}_n$, with $n \geq 2$, i.c.c. hyperbolic groups, non-trivial free products $\Gamma_1 * \Gamma_2$, certain groups with positive first ℓ^2-Betti number, etc. For the precise statement we refer to Theorem 1.2 below.

In order to state our main result, we first need to introduce a few notions. Recall from [CH88] that a countable group Γ is said to be *weakly amenable* if it admits a sequence of finitely supported functions $\varphi_n : \Gamma \to \mathbb{C}$ tending to 1 pointwise and satisfying $\sup_n \|\varphi_n\|_{\mathrm{cb}} < \infty$, where $\|\varphi\|_{\mathrm{cb}}$ denotes the Herz-Schur norm of φ (i.e. the cb-norm of the linear map $\mathrm{L}\Gamma \ni u_g \mapsto \varphi(g)u_g \in \mathrm{L}\Gamma$).

If Γ is a countable group and $\pi : \Gamma \to \mathcal{O}(\mathcal{K}_\mathbb{R})$ is an orthogonal representation of Γ on a real Hilbert space $\mathcal{K}_\mathbb{R}$, then a *1-cocycle* c into π is a map $c : \Gamma \to \mathcal{K}_\mathbb{R}$ satisfying the cocycle relation $c(gh) = c(g) + \pi(g)c(h),$ for all $g, h \in \Gamma$.

A subgroup $\Sigma < \Gamma$ is called *malnormal* if $\Sigma \cap g\Sigma g^{-1} = \{1\}$, for all $g \in \Gamma \setminus \Sigma$. If $\{\Sigma_i\}_{i\in I}$ is a family of subgroups of Γ, then we say that $\{\Sigma_i\}_{i\in I}$ is *malnormal* in Γ if $g\Sigma_i g^{-1} \cap \Sigma_j = \{1\}$, unless $i = j$ and $g \in \Sigma_i$.

If Γ is a countable group, $\Sigma < \Gamma$ is a subgroup and $\theta : \Sigma \to \Gamma$ is an injective group homomorphism, then the HNN extension $\mathrm{HNN}(\Gamma, \Sigma, \theta)$ is the group generated by a copy of Γ and an extra generator t, called stable letter, subject to relations $tgt^{-1} = \theta(g)$, for all $g \in \Sigma$. We say that $\mathrm{HNN}(\Gamma, \Sigma, \theta)$ is *non-degenerate* if $\Sigma \neq \Gamma \neq \theta(\Sigma)$. Note that, in this case, $\mathrm{HNN}(\Gamma, \Sigma, \theta)$ contains a copy of the free group on two generators, hence it is non-amenable. In the same spirit, we say that an amalgamated free product $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ is *non-degenerate* if $[\Gamma_1 : \Sigma] \geq 2$ and $[\Gamma_2 : \Sigma] \geq 3$, and this is sufficient to witness the non-amenability of Γ.

Theorem 1.2. *Assume that Γ is one of the following groups:*

(a) an i.c.c. non-elementary hyperbolic group,

(b) a finitely generated, i.c.c., non-amenable, discrete subgroup of a connected non-compact rank one simple Lie group with finite center,

(c) a non-degenerate amalgamated free product $\Gamma_1 *_\Sigma \Gamma_2$ or a non-degenerate HNN extension HNN$(\Gamma_1, \Sigma, \theta)$ with Σ malnormal in Γ_1, respectively $\{\Sigma, \theta(\Sigma)\}$ malnormal in Γ_1,

(d) an i.c.c., weakly amenable group with positive first ℓ^2-Betti number and that admits a bound on the orders of its finite subgroups.

Then all of the following generalized wreath product groups $\mathcal{G}$ are W-superrigid in the sense of Definition 1.1:*

1. the group $(\mathbb{Z}/n\mathbb{Z})^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$ where $n \in \{2, 3\}$,

2. *the kernel of the homomorphism* $H^{(\Gamma)} \rtimes (\Gamma \times \Gamma) \to H : xg \mapsto \sum_{k \in \Gamma} x_k$*, where* H *is an arbitrary non-trivial torsion-free abelian group.*

It is worth to remark that there exist non-amenable i.c.c. groups Γ such that $\mathrm{L}\mathcal{G}$ is a *McDuff factor*, i.e. $\mathrm{L}\mathcal{G} \cong \mathrm{L}\mathcal{G} \mathbin{\overline{\otimes}} \mathcal{R}$, where $\mathcal{R}$ is the hyperfinite II_1 factor. Thus not all non-amenable left-right wreath products are W*-superrigid.

By [BV97], [PT07], a countable group Γ has positive *first ℓ^2-Betti number* if and only if it is non-amenable and it admits an unbounded 1-cocycle into the left regular representation. Actually, throughout this thesis we only use this characterization of having positive first ℓ^2-Betti number, without defining explicitly ℓ^2-Betti numbers for countable groups. In [PT07, Section 3], there are given many examples of countable groups Γ with positive first ℓ^2-Betti number, such as certain amalgamated free products, certain HNN extensions, hyperbolic triangle groups, limit groups of Sela, etc. Moreover, [PT07, Theorem 3.2] provides a very useful formula for estimating from below the first ℓ^2-Betti number of a group defined by (a finite number of) generators and relations. It is known that all *Coxeter groups* are weakly amenable [Ja98],[Val93]. Using [PT07, Theorem 3.2] one can construct Coxeter groups with positive first ℓ^2-Betti number (see, for instance, [KN11]).

Remark 1.3. Let Γ be a group as in Theorem 1.2. Assume moreover that Γ has no non-trivial characters. Let H be an arbitrary non-trivial torsion-free abelian group and denote by $\mathcal{G}_0$ the kernel of the homomorphism $H^{(\Gamma)} \rtimes (\Gamma \times \Gamma) \to H$ given in Theorem 1.2. At the end of Chapter 7, we prove that $\mathcal{G}_0$ has no characters either. So the conclusion of Theorem 1.2 becomes stronger: whenever Λ is a countable group and $\pi : L\Lambda \to (L\mathcal{G}_0)^r$ is a $*$-isomorphism, we have $r = 1$ and there exist an isomorphism of groups $\delta : \Lambda \to \mathcal{G}_0$ and a unitary $w \in L(\mathcal{G}_0)$ such that $\pi(v_s) = w\, u_{\delta(s)}\, w^*$ for all $s \in \Lambda$.

Comments on the proofs

To describe more precisely the general strategy of [IPV10], let $G = H \wr_I \Gamma$ be a generalized wreath product as above. Write $M := \mathrm{L}G$ and assume that $M \cong \mathrm{L}\Lambda$, for an arbitrary countable group Λ. Denote by $\Delta_\Lambda : M \to M \mathbin{\overline{\otimes}} M$ the comultiplication corresponding to the decomposition $M \cong \mathrm{L}\Lambda$, defined by $\Delta(v_s) = v_s \otimes v_s$, for all $s \in \Lambda$. Similarly, define $\Delta_G : M \to M \mathbin{\overline{\otimes}} M$. Observe that M can be written as the group measure space von Neumann algebra $M = \mathrm{L}^\infty(X^I) \rtimes \Gamma$, where $X = \widehat{H}$ is the dual of H equipped with the Haar probability measure and where $\Gamma \curvearrowright X^I$ is the generalized Bernoulli action. The general methods of [Io10], extended to generalized Bernoulli actions in [IPV10], allow us to classify all possible embeddings $\Delta : M \to M \mathbin{\overline{\otimes}} M$. When applying this classification to Δ_Λ, we deduce the existence of a unitary element $\Omega \in M \mathbin{\overline{\otimes}} M$ such that $\Delta_\Lambda(x) = \Omega \Delta_{\mathcal{G}}(x)\Omega^*$, for all $x \in M$. Moreover, this unitary Ω satisfies a certain "dual" 2-cocycle relation. The last step consists of a vanishing result of the 2-cohomology, saying that any such dual 2-cocycle Ω should vanish and this allows us to conclude that G and Λ are isomorphic.

To prove our W*-superrigidity Theorem 1.2, we follow the approach of [IPV10] described above. Let $\mathcal{G} := (\mathbb{Z}/2\mathbb{Z})^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$ be a left-right wreath product as in Theorem 1.2.1 and denote $M := \mathrm{L}\mathcal{G}$. We consider the comultiplication $\Delta : \mathrm{L}\Lambda \to \mathrm{L}\Lambda \mathbin{\overline{\otimes}} \mathrm{L}\Lambda$ that is induced by another group von Neumann algebra decomposition $M = \mathrm{L}\Lambda$ and we carefully analyze how Δ relates to the initial von Neumann algebra structure of $M = L\mathcal{G}$. The following are the two major steps in the proof. We first use Popa's malleable deformation for Bernoulli actions [Po03] and his spectral gap rigidity [Po06b] to prove that the subalgebra $\mathrm{L}(\Gamma \times \Gamma) \subset \mathrm{L}\mathcal{G}$ is invariant under Δ, up to unitary conjugacy. We next use the recent results on normalizers of amenable subalgebras from [PV11], [PV12], [Io12b] and [Va13] to prove that also the subalgebra $\mathrm{L}\big((\mathbb{Z}/2\mathbb{Z})^{(\Gamma)}\big) \subset \mathrm{L}\mathcal{G}$ is invariant under Δ, up to unitary conjugacy. Both steps together bring us to a point where the general strategy of [IPV10] can be applied. Contrary to the approach of [IPV10], our proof does not use the *clustering techniques* of [Po04]. As a consequence, we can also prove W*-superrigidity for a number of subgroups of generalized wreath product groups (see Theorem 1.2.2).

Chapter 2

Preliminaries

2.1 Von Neumann algebras

2.1.1 Classification of factors into types

Let $\mathcal{H}$ be a complex Hilbert space and denote by $B(\mathcal{H})$ the algebra of all bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$. We work throughout this thesis only with separable Hilbert spaces.

A *von Neumann algebra* M is a $*$-subalgebra of $B(\mathcal{H})$ that is closed in the strong operator topology and contains the unit of $B(\mathcal{H})$. The celebrated theorem of von Neumann [vN29] gives a purely algebraic characterization of a von Neumann algebra, in the following sense. If $\mathcal{F} \subset B(\mathcal{H})$ is a subset, then we denote by $\mathcal{F}'$ its commutant in $B(\mathcal{H})$, i.e. the set consisting of all operators in $B(\mathcal{H})$ that commute with all operators in $\mathcal{F}$. If $\mathcal{F} \subset B(\mathcal{H})$ is self-adjoint, then clearly $\mathcal{F}'$ is self-adjoint and strongly closed and hence it is a von Neumann algebra. If $M \subset B(\mathcal{H})$ is a unital $*$-subalgebra, then *von Neumann's bicommutant theorem* says that M is a von Neumann algebra if and only if it coincides with its double commutant M''. Remark that there is also an abstract approach to von Neumann algebras which does not rely on any concrete Hilbert space. As shown by Sakai [Sa56], a von Neumann algebra can be defined abstractly as a C*-algebra that is the dual of some Banach space, called *predual*. Moreover, this predual is unique and every abstract von Neumann algebra with separable predual can be realized as a concrete von Neumann algebra on a separable Hilbert space.

The simplest example of von Neumann algebra is of course $B(\mathcal{H})$. Every standard probability measure space (X, μ) gives rise to an abelian von Neumann algebra $\mathrm{L}^\infty(X, \mu)$ acting on the Hilbert space $\mathrm{L}^2(X, \mu)$ by multiplication. The first non-trivial examples that appeared in the work of Murray and von Neumann [MvN36], [MvN43] were von Neumann algebras associated to countable groups and to actions of countable groups on probability measure spaces. Before studying these examples in more details we will restrict to the special case when our von Neumann algebras have trivial center (for instance, $B(\mathcal{H})$ has trivial center). A von Neumann algebra whose center consists only of scalar multiples of the identity is called *factor*. In [vN49] is shown that any von Neumann algebra can be decomposed as a direct integral of factors, thus it suffices to study and understand factors. But even the structure of von Neumann factors is still far from being well understood.

In [MvN36], von Neumann factors are classified into three types: I, II and III. Before making this classification more precise let us introduce some terminology. Let M be a von Neumann algebra and let $\varphi : M \to \mathbb{C}$ be a linear functional. The functional φ is called *positive* if $\varphi(x^*x) \geq 0$, for all $x \in M$, and *faithful* if $\varphi(x^*x) = 0$ implies that $x = 0$. We say that φ is a *state* if it is positive and $\varphi(1) = 1$. A state φ is called *normal* if it is continuous for the ultraweak topology on M (or equivalently, its restriction to the unit ball of M is weakly continuous). If $\varphi(xy) = \varphi(yx)$, for all $x, y \in M$, then we say that φ is a *tracial state* or a *trace* on M. Any positive functional φ on M satisfies the Cauchy-Schwarz inequality, namely $|\varphi(xy)|^2 \leq \varphi(x^*x) \cdot \varphi(y^*y)$, for all $x, y \in M$.

A *tracial von Neumann algebra* (M, τ) is a von Neumann algebra M equipped with a normal faithful tracial state τ. Any tracial von Neumann algebra (M, τ) admits a standard representation on a Hilbert space denoted by $\mathrm{L}^2(M, \tau)$, constructed in the following way. Define a sesquilinear form on M by the formula $\langle x, y\rangle_\tau := \tau(y^*x)$, for all $x, y \in M$. Denote by $\mathrm{L}^2(M, \tau)$, or simply $\mathrm{L}^2(M)$, the completion of M with respect to this sesquilinear form and by $\|\cdot\|_2$ the corresponding norm defined by $\|x\|_2 = \tau(x^*x)^{1/2}$, for all $x \in M$. Write $M \ni x \mapsto x\hat{1} = \hat{x} \in \mathrm{L}^2(M)$ for the natural embedding and notice that the vector $\hat{1} \in \mathrm{L}^2(M)$ is cyclic and separating for M and $\|xy\|_2 \leq \|x\| \cdot \|y\|_2$, for all $x, y \in M$. Thus we can represent M in a standard way on $\mathrm{L}^2(M)$ by letting $\pi(x)\hat{y} = \widehat{xy}$, for all $x, y \in M$. The representation π is called the *GNS representation* of (M, τ) and it is a normal $*$-representation. We will usually identify M with its image and see $M \subset B(\mathrm{L}^2(M))$.

Let M be an arbitrary von Neumann algebra and denote by M^+ the positive cone of M. A *weight* ψ on M is a linear functional ψ from the positive cone M^+ to $[0, \infty]$. We say that ψ is *semifinite* if the linear span of $\{x \in M^+ \mid \psi(x) < \infty\}$ is ultraweakly dense in M^+. We also say that ψ is *faithful* if $\psi(x) = 0$ implies that $x = 0$ and that ψ is *normal* if for any bounded increasing sequence (x_n)

in M^+ we have that $\psi(\sup_n x_n) = \sup_n \psi(x_n)$ (or equivalently, if ψ is lower ultraweakly semicontinuous). Notice that any von Neumann algebra has a faithful semifinite normal weight. If $\psi(x^*x) = \psi(xx^*)$, for all $x \in M$, then we say that ψ is a *tracial weight* or simply a *trace* on M.

We are now ready to make the classification of von Neumann factors precise. If M is a factor, then we say that:

- M is a type I factor if and only if $M \cong B(\mathcal{H})$, for some Hilbert space $\mathcal{H}$;
- M is a type II_1 factor if and only if M is infinite dimensional and admits a finite normal tracial state;
- M is a type II_∞ factor if and only if $M \ncong B(\mathcal{H})$, for any Hilbert space $\mathcal{H}$, and it admits a normal semifinite tracial weight Tr such that $\mathrm{Tr}(1) = \infty$;
- M is a type III factor if and only if any normal trace on M is zero.

In this thesis we will be mainly focusing on type II_1 factors and their classification. The main sources of examples of II_1 factors are countable groups and their actions on probability measure spaces. Before going to study these constructions, let us introduce a few more general notions.

Let (M, τ) be a tracial von Neumann algebra and $A \subset M$ be a von Neumann subalgebra. We say that A is *maximal abelian* in M if $A' \cap M = A$ and that A is *regular* in M if the normalizer $\mathcal{N}_M(A) := \{u \in \mathcal{U}(M) \mid uAu^* = A\}$ of A in M generates M as a von Neumann algebra. If A is maximal abelian and regular in M, then we say that A is a *Cartan subalgbera* of M.

Let (M, Tr) be a von Neumann algebra with a normal semifinite trace Tr. Consider the following two subspaces $N_{\mathrm{Tr}} := \{x \in M \mid \mathrm{Tr}(x^*x) < \infty\}$ and $M_{\mathrm{Tr}} := N_{\mathrm{Tr}} \cdot N_{\mathrm{Tr}}$ and for all $x \in M_{\mathrm{Tr}}$, define $\|x\|_1 := \mathrm{Tr}(|x|)$. Then $(M_{\mathrm{Tr}}, \|\cdot\|_1)$ becomes a normed space and we denote by $\mathrm{L}^1(M, \mathrm{Tr})$ the Banach space completion of M_{Tr} with respect to the $\|\cdot\|_1$-norm. One can prove that whenever φ is a normal positive form on M, there exists a positive element $T \in \mathrm{L}^1(M, \mathrm{Tr})^+$ such that $\varphi(x) = \mathrm{Tr}(Tx)$, for all $x \in M$. In other words, $\mathrm{L}^1(M, \mathrm{Tr})$ is a Banach space whose dual is M.

2.1.2 The group von Neumann algebra

To any countable group Γ we can associate the *group von Neumann algebra* $\mathrm{L}\Gamma$, generated by the image of the left regular representation of Γ on the Hilbert space $\ell^2(\Gamma)$. This construction is due to Murray and von Neumann [MvN36], where they use it to give the first example of a factor different from $B(\mathcal{H})$.

Let $\{\delta_g\}_{g\in\Gamma}$ be the canonical orthonormal basis of $\ell^2(\Gamma)$. Then the left regular representation λ of Γ on the Hilbert space $\ell^2(\Gamma)$ is defined by

$$\lambda(g)(\delta_h) = \delta_{gh}, \text{ for all } g, h \in \Gamma.$$

We define the group von Neumann algebra $\mathrm{L}\Gamma$ as the von Neumann algebra generated by the group of unitaries $\{u_g := \lambda(g) \mid g \in \Gamma\} \subset B(\ell^2(\Gamma))$. For any countable group Γ, the group von Neumann algebra $\mathrm{L}\Gamma$ has a normal faithful tracial state τ given by $\tau(x) = \langle \delta_e, x\delta_e\rangle$, for all $x \in \mathrm{L}\Gamma$. Remark also that $\mathrm{L}\Gamma$ is already on standard form on the Hilbert space $\ell^2(\Gamma)$.

One can easily check that $\mathrm{L}\Gamma$ is a II_1 factor if and only if Γ is an *i.c.c.* group, i.e. the conjugacy classes of all non-trivial elements in Γ are infinite. Examples of i.c.c. groups are the free groups $\mathbb{F}_n$, the infinite symmetric group S_∞, the projective special linear groups $\mathrm{PSL}(n, \mathbb{Z})$, with $n \geq 2$, etc.

If Γ is an abelian group, then its dual $\widehat{\Gamma}$ is a second countable compact group and the Fourier transform $\ell^2(\Gamma) \to L^2(\widehat{\Gamma})$ defines a canonical identification $\mathrm{L}\Gamma \cong \mathrm{L}^\infty(\widehat{\Gamma})$.

2.1.3 Amenability, hyperfiniteness and property Gamma

Let (M, τ) be a tracial von Neumann algebra and denote $\mathcal{H} := \mathrm{L}^2(M)$. If $B \subset M$ is a von Neumann subalgebra, then there exists a unique trace-preserving faithful normal *conditional expectation* from M onto B, i.e. a unital completely positive B-B-bimodular map $E_B : M \to B$. If e_B denotes the orthogonal projection from $\mathrm{L}^2(M)$ onto $\mathrm{L}^2(B)$, then we have that $e_B(\hat{x}) = \widehat{E_B(x)}$, for all $x \in M$.

The von Neumann algebra (M, τ) is called *injective* if there exists a conditional expectation from $B(\mathcal{H})$ onto M and *amenable* if there exists an M-central state φ on $B(\mathcal{H})$ whose restriction to M equals τ. Such a state is called an *M-hypertrace* on $B(\mathcal{H})$. Here, the M-centrality of φ means that $\varphi \circ \mathrm{Ad}(u) = \varphi$, for every unitary $u \in \mathcal{U}(M)$. Recall that a countable group Γ is amenable if there exists a Γ-invariant state on $\ell^\infty(\Gamma)$. One can prove that Γ is amenable if and only if the group von Neumann algebra $\mathrm{L}\Gamma$ is amenable.

In [Co76], Connes proved that injectivity and amenability are equivalent and, moreover, that both are equivalent with being *approximately finite dimensional* or *hyperfinite*. A von Neumann algebra M is called hyperfinite if there exists an increasing sequence of finite dimensional subalgebras such that their union generates M as a von Neumann algebra.

Murray and von Neumann [MvN43] proved the existence of a unique *hyperfinite* II_1 factor $\mathcal{R}$, up to isomorphism, defined as the bicommutant of an increasing union of matrix algebras, or equivalently, as the group von Neumann algebra of S_∞. Thus, Connes' famous result implies that there is a unique amenable II_1 factor, up to isomorphism. In other words, the hyperfinite factors are isolated among all II_1 factors by amenability (or injectivity). Since $\mathrm{L}\Gamma$ is amenable if and only if the group Γ is amenable, it follows that all the group von Neumann algebras arising from i.c.c. amenable groups are isomorphic.

Distinguishing between group factors is usually a very hard problem. Murray and von Neumann [MvN43] proved that the II_1 factors $\mathrm{L}S_\infty$ and $\mathrm{L}\mathbb{F}_2$ are non-isomorphic, using an invariant called *property Gamma*. Very few non-isomorphic II_1 factors were discovered until 1969 when McDuff constructed an uncountable family of non-isomorphic II_1 factors in [McD69]. As we have seen in the introduction, it is unknown whether the free group factors $\mathrm{L}\mathbb{F}_n$ and $\mathrm{L}\mathbb{F}_m$ are isomorphic or not, for different n and m.

A bounded sequence (a_n) in a II_1 factor M is said to be *central* if

$$\lim_n \|a_n x - x a_n\|_2 = 0, \text{ for all } x \in M.$$

A central sequence (a_n) is called *trivial* if $\lim_n \|a_n - \tau(a_n)1\|_2 = 0$. The II_1 factor M is said to have *property Gamma,* if M admits a non-trivial central sequence. Property Gamma is equivalent with the existence of a central sequence of unitaries $u_n \in M$ with $\tau(u_n) = 0$ for all n.

A closely related notion for groups is the inner amenability. A countable group Γ is *inner amenable* if the unitary representation $(\mathrm{Ad}\, g)_{g\in\Gamma}$ on $\ell^2(\Gamma) \ominus \mathbb{C}$ has almost invariant vectors: there exists a sequence of unit vectors $\xi_n \in \ell^2(\Gamma) \ominus \mathbb{C}$ such that $\lim_n \|(\mathrm{Ad}\, g)(\xi_n) - \xi_n\|_2 = 0$, for every $g \in \Gamma$. Examples of inner amenable groups are amenable groups, direct products $G \times H$, with H infinite amenable, Baumslag-Solitar groups, etc.

Let Γ be an i.c.c. group. By [Ef73], if $\mathrm{L}\Gamma$ has property Gamma, then Γ must be inner amenable. The converse can however fail, as was shown in [Va09].

Denote by $\mathcal{R}$ the unique hyperfinite II_1 factor. A II_1 factor M is said to be *McDuff* if M is isomorphic with $M \mathbin{\overline{\otimes}} \mathcal{R}$. One can prove that every McDuff II_1 factor has property Gamma. By [McD70], a II_1 factor M is McDuff if and only if M admits two central sequences of unitaries $u_n, v_n \in M$ such that $\tau(u_n) = \tau(v_n) = \tau(u_n v_n u_n^* v_n^*) = 0$ for all n.

For any von Neumann algebra M (with separable predual M_*), we denote by $\mathrm{Aut}(M)$ the group of automorphisms of M, with its natural topology given by the following seminorms $\mathrm{Aut}(M) \ni \varphi \mapsto \|\theta \circ \varphi\| \in \mathbb{C}$, for every $\theta \in M_*$. Since the predual of M is separable, then $\mathrm{Aut}(M)$, with this topology, is a Polish group. We denote by $\mathrm{Inn}(M)$ the normal subgroup of inner automorphisms $\mathrm{Ad}(u)$, $u \in \mathcal{U}(M)$, and by $\mathrm{Out}(M) := \mathrm{Aut}(M)/\mathrm{Inn}(M)$ the quotient group.

Then M is non-Gamma if and only if $\mathrm{Inn}(M)$ is closed in $\mathrm{Aut}(M)$. In that case, $\mathrm{Out}(M)$ naturally becomes a Polish group as well.

2.1.4 The group measure space construction

Let Γ be a countable group and let $\Gamma \curvearrowright (X,\mu)$ be a probability measure preserving action (p.m.p.) of Γ on a standard probability measure space (X,μ). Using these data we can define a von Neumann algebra $\mathrm{L}^\infty(X) \rtimes \Gamma$ called the *group measure space construction* of Murray and von Neumann [MvN43]. More generally, given any trace-preserving action of Γ on a tracial von Neumann algbera (B,τ), we can define the *crossed product* $B \rtimes \Gamma$.

Let $\Gamma \curvearrowright (X,\mu)$ be a p.m.p. action. This induces an action $\sigma : \Gamma \curvearrowright \mathrm{L}^\infty(X)$, defined by $\sigma_g(f)(x) = f(g^{-1}x)$, for all $f \in \mathrm{L}^\infty(X)$ and $x \in X$, $g \in \Gamma$. We still denote by σ the corresponding Koopman representation $\sigma : \Gamma \to \mathcal{U}(\mathrm{L}^2(X))$. Consider the Hilbert space $\mathcal{H} := \mathrm{L}^2(X) \otimes \ell^2(\Gamma)$ and denote by λ the left regular representation of Γ. Clearly, one can see $\mathrm{L}^\infty(X) = \mathrm{L}^\infty(X) \mathbin{\overline{\otimes}} 1 \subset B(\mathcal{H})$. By Fell's absorption principle, the unitary representation $u_g := \sigma_g \otimes \lambda_g$ of Γ on $\mathcal{H}$ is just a multiple of λ and moreover it satisfies the following covariance relation: $u_g f u_g^* = \sigma_g(f)$, for all $g \in \Gamma$ and $f \in \mathrm{L}^\infty(X)$. Thus, we can define the group measure space construction $\mathrm{L}^\infty(X) \rtimes \Gamma$ as the von Neumann algebra

$$\mathrm{L}^\infty(X) \rtimes \Gamma := \left\{ \sum_{\text{finite}} a_g u_g \mid a_g \in \mathrm{L}^\infty(X) \right\}'' \subset B(\mathcal{H}),$$

with the faithful normal tracial state τ given by $\tau(x) = \langle x(1 \otimes \delta_e), 1 \otimes \delta_e \rangle$, for all $x \in \mathrm{L}^\infty(X) \rtimes \Gamma$.

Recall that the action $\Gamma \curvearrowright (X,\mu)$ is *essentially free* if $\mu(\{x \in X \mid gx = x\}) = 0$, for any $g \neq e$, and *ergodic* if any Γ-invariant subset $A \subset X$ is either null or conull. One can prove that $\Gamma \curvearrowright (X,\mu)$ is essentially free if and only if $\mathrm{L}^\infty(X)$ is maximal abelian in $\mathrm{L}^\infty(X) \rtimes \Gamma$. In particular, $\mathrm{L}^\infty(X)$ is a Cartan subalgebra of $\mathrm{L}^\infty(X) \rtimes \Gamma$. Moreover, under the freeness assumption, we have that $\Gamma \curvearrowright (X,\mu)$ is ergodic if and only if $\mathrm{L}^\infty(X) \rtimes \Gamma$ is a II_1 factor.

Every element $x \in \mathrm{L}^\infty(X) \rtimes \Gamma$ admits a unique $\|\cdot\|_2$-norm *Fourier decomposition* $x = \sum_{g \in \Gamma} x_g u_g$, with $x_g := E_{\mathrm{L}^\infty(X)}(x u_g^*) \in \mathrm{L}^\infty(X)$, for all $g \in \Gamma$. Moreover, $\|x\|_2^2 = \sum_{g \in \Gamma} \|x_g\|_2^2$.

A similar construction can be performed for any trace-preserving action of Γ on a tracial von Neumann algebra (B,τ). Let $\sigma : \Gamma \curvearrowright (B,\tau)$ be an action by trace-preserving automorphisms $\sigma_g \in \mathrm{Aut}(B)$. Then the crossed product $B \rtimes \Gamma$ is the unique tracial von Neumann algebra (M,τ) generated by a trace-preserving copy of B and unitary elements $(u_g)_{g\in\Gamma}$ satisfying the following relations: $u_g b u_g^* = \sigma_g(b)$, for all $b \in B$, $g \in \Gamma$, $u_g u_h = u_{gh}$, for all $g, h \in \Gamma$, and $\tau(bu_g) = 0$, for all $b \in B$ and $g \neq e$. Notice that the map $bu_g \mapsto b \otimes \delta_g$ provides an identification $\mathrm{L}^2(M) \cong \mathrm{L}^2(B) \otimes \ell^2(\Gamma)$, and hence an explicit realization of M as a von Neumann algebra on the Hilbert space $\mathrm{L}^2(B) \otimes \ell^2(\Gamma)$, as in the group measure space construction case.

2.1.5 Jones' basic construction

Let (M, τ) be a tracial von Neumann algebra and $Q \subset M$ be a von Neumann subalgebra. The *Jones' basic construction* for the inclusion $Q \subset M$ is defined as the von Neumann algebra $\langle M, e_Q \rangle$ generated by M and the orthogonal projection $e_Q : \mathrm{L}^2(M) \to \mathrm{L}^2(Q)$.

We list now the main properties of the basic construction. Denote by Me_QM the linear span of the set $\{xe_Qy \mid x, y \in M\}$.

If Q is a von Neumann subalgebra of a tracial von Neumann algebra (M, τ), then the basic construction $\langle M, e_Q \rangle$ is a semifinite von Neumann algebra with a faithful normal semifinite trace Tr satisfying the following properties:

- $\langle M, e_Q \rangle$ equals the commutant of the right action of Q on $\mathrm{L}^2(M)$ and the $*$-subalgebra Me_QM is weakly dense in $\langle M, e_Q \rangle$;
- $\mathrm{Tr}(xe_Qy) = \tau(xy)$, for all $x, y \in M$;
- $e_Qxe_Q = E_Q(x)e_Q = e_QE_Q(x)$, for all $x \in M$;
- the central support of e_Q in $\langle M, e_Q \rangle$ is 1;
- Me_QM is dense in $\mathrm{L}^2(\langle M, e_Q \rangle)$ in $\|\cdot\|_{2,\mathrm{Tr}}$-norm.

Part of these properties characterize the basic construction, as in the following well-known result (see e.g. [SS08, Theorem 3.3.15]).

Theorem 2.1. *Let N be a semifinite von Neumann algebra with a faithful normal semifinite trace* Tr *and von Neumann subalgebras $Q \subset M \subset N$. Assume that $e \in N$ is a projection such that*

1. *N is the weak closure of the $*$-subalgebra MeM;*
2. *$Tr(e) = 1$ and $\tau(x) := \mathrm{Tr}(xe)$ defines a faithful normal trace τ on M;*
3. *$eNe = Qe = eQ$;*

Then there is a trace-preserving $*$-isomorphism $\theta : \langle M, e_Q\rangle \to N$ with $\theta(x) = x$, for all $x \in M$, and $\theta(e_Q) = e$.

Lemma 2.2. *Let $\Gamma \curvearrowright (B,\tau)$ be a trace-preserving action of a countable group Γ on a tracial von Neumann algebra (B,τ). Let $\Sigma < \Gamma$ be a subgroup and denote $M := B \rtimes \Gamma$ and $Q := B \rtimes \Sigma$. Then the basic construction $\langle M, e_Q\rangle$ is isomorphic to $N = (B \overline{\otimes} \ell^\infty(\Gamma/\Sigma)) \rtimes \Gamma$, where Γ acts on $B \overline{\otimes} \ell^\infty(\Gamma/\Sigma)$ diagonally.*

Proof. Define the projection $e := 1 \otimes \delta_{e\Sigma} \in N$ and note that we can see Q and M as subalgebras of the semifinite von Neumann algebra $N = (B \overline{\otimes} \ell^\infty(\Gamma/\Sigma)) \rtimes \Gamma \cong (B \rtimes \Sigma) \overline{\otimes} B(\ell^2(\Gamma/\Sigma))$. One can easily check that $Q \subset M \subset N$ and e satisfy all assumptions of Lemma 2.1 and hence the basic construction $\langle M, e_Q\rangle$ is isomorphic to N. $\square$

2.2 Bimodules and weak containment

Let (M,τ) be a tracial von Neumann algebra. A *left M-module* ${}_M\mathcal{H}$ is a Hilbert space $\mathcal{H}$ equipped with a normal unital $*$-homomorphism $\mathrm{M} \to B(\mathcal{H})$. For any $x \in M$ and $\xi \in \mathcal{H}$, we denote the action of x on ξ simply by $x\xi$. If M^{op} denotes the opposite von Neumann algebra of M, then we may define the similar notion of right M-module $\mathcal{H}_M$ as a Hilbert space $\mathcal{H}$ equipped with a normal unital $*$-homomorphism $M^{\mathrm{op}} \to B(\mathcal{H})$. All Hilbert modules considered in this thesis are assumed to be separable.

The most obvious example of left/right M-module is the GNS Hilbert space $\mathrm{L}^2(M)$. Recall that M acts from the left and from the right on $\mathrm{L}^2(M)$ and these two actions commute inside $B(\mathrm{L}^2(M))$. Further, if we denote by $\ell^2(\mathbb{N}) \otimes \mathrm{L}^2(M)$ the direct sum of countably many copies of $\mathrm{L}^2(M)$, then this is also a left/right M-module.

In general, if $\mathcal{H}_M$ is any right M-module, then one can prove that $\mathcal{H}$ is isomorphic, as a right M-module, to $p(\ell^2(\mathbb{N}) \otimes \mathrm{L}^2(M))_M$, for some projection $p \in B(\ell^2(\mathbb{N})) \overline{\otimes} M$. This precisely means that there exists an isometry $v : \mathcal{H} \to \ell^2(\mathbb{N}) \otimes \mathrm{L}^2(M)$ with $vv^* = p$ and such that $v(\xi x) = v(\xi)x$, for all $x \in M$ and $\xi \in \mathcal{H}$. If there is another projection $q \in B(\ell^2(\mathbb{N})) \overline{\otimes} M$ such that $\mathcal{H}_M$ is isomorphic to $q(\ell^2(\mathbb{N}) \otimes \mathrm{L}^2(M))_M$, then q must be equivalent to p. Therefore, it makes sense to consider the number $\dim(\mathcal{H}_M) := (\mathrm{Tr} \otimes \tau)(p) \in [0,\infty]$ and call it the (right) *M-dimension* of $\mathcal{H}$. Here, Tr denotes the canonical trace on $B(\ell^2(\mathbb{N}))$. As we just remarked, $\dim(\mathcal{H}_M)$ does not depend on the projection p, but it depends on the trace τ. The M-dimension of a left M-module can be defined similarly.

Let M and N be tracial von Neumann algebras. An *M-N-bimodule* ${}_M\mathcal{H}_N$ is a Hilbert space $\mathcal{H}$ equipped with two commuting normal unital $*$-homomorphisms $M \to B(\mathcal{H})$ and $N^{\mathrm{op}} \to B(\mathcal{H})$. As we already noticed before, $\mathrm{L}^2(M)$ is obviously an M-M-bimodule, called the *standard* or the *trivial* M-M-bimodule. Note that, up to isomorphism, it does not depend on the choice of the normal faithful trace on M. The tensor product $\mathrm{L}^2(M) \otimes \mathrm{L}^2(N)$ is also an M-N-bimodule with the bimodule action given by $x(\xi \otimes \eta)y = x\xi \otimes \eta y$, for all $x \in M$, $y \in N$, $\xi \in \mathrm{L}^2(M)$ and $\eta \in \mathrm{L}^2(N)$. If $M = N$, then the M-M-bimodule $\mathrm{L}^2(M)\otimes\mathrm{L}^2(M)$ is called the *coarse* M-M-bimodule.

Another source of examples of bimodules is given by the unitary representations of a countable group. Let Γ be a countable group and denote by M its group von Neumann algebra $\mathrm{L}\Gamma$. Recall that, in this case, $\mathrm{L}^2(M) = \ell^2(\Gamma)$ and the M-M-bimodule structure of $\mathrm{L}^2(M)$ is given by the formula $u_g \xi u_h^* = \lambda(g)\rho(h)\xi$, for all $g, h \in \Gamma$ and $\xi \in \ell^2(\Gamma)$. Here λ and ρ denote the left, respectively the right regular representation of Γ. Let $(\pi, \mathcal{H})$ be any unitary representation of Γ. Then we may define an M-M-bimodule ${}_M\mathcal{H}^\pi{}_M$ by $\mathcal{H}^\pi = \ell^2(\Gamma) \otimes \mathcal{H}$ and $u_g(\xi \otimes \eta)u_h^* = (u_g \xi u_h^*) \otimes \pi(g)\eta$, for all $g, h \in \Gamma$, $\xi \in \ell^2(\Gamma)$ and $\eta \in \mathcal{H}$. The right action is just $\rho \otimes 1$ and it clearly extends to M. The left action is given by $\lambda \otimes \pi$, and by Fell's absorption principle, it is equivalent to a multiple of λ and hence it also extends to M and commutes with the right action. Note that the isomorphism class of $\mathcal{H}^\pi$ depends only on the equivalence class of the representation π. One can see immediately that the trivial M-M-bimodule $\ell^2(\Gamma)$ corresponds to the trivial representation of Γ and the coarse M-M-bimodule $\ell^2(\Gamma) \otimes \ell^2(\Gamma)$ corresponds to the left regular representation of Γ. This construction can be easily extended to crossed products, as follows: if $\Gamma \curvearrowright (B, \tau)$ is a trace-preserving action and $M = B \rtimes \Gamma$ and if $(\pi, \mathcal{H})$ is a unitary representation of Γ, then $\mathrm{L}^2(M) = \ell^2(\Gamma) \otimes \mathrm{L}^2(B)$ and we define $\mathcal{H}^\pi = \mathrm{L}^2(M) \otimes \mathcal{H}$, with the obvious actions.

If $(\rho, \mathcal{K})$ and $(\pi, \mathcal{H})$ are unitary representations of Γ, we say that ρ is *weakly contained* in π and we write $\rho \prec \pi$ if $\|\rho(a)\| \leq \|\pi(a)\|$ for all $a \in \mathbb{C}\Gamma$.

Any bimodule ${}_M\mathcal{H}_N$ gives rise to a $*$-homomorphism $\pi_{\mathcal{H}} : M \otimes_{\mathrm{alg}} N^{\mathrm{op}} \to B(\mathcal{H})$ given by $\pi_{\mathcal{H}}(x \otimes y^{\mathrm{op}})\xi = x\xi y$, for all $x \in M$, $y \in N$ and $\xi \in \mathcal{H}$.

Similarly, if ${}_M\mathcal{K}_N$ and ${}_M\mathcal{H}_N$ are M-N-bimodules, we say that ${}_M\mathcal{K}_N$ is *weakly contained* in ${}_M\mathcal{H}_N$ and we write ${}_M\mathcal{K}_N \prec {}_M\mathcal{H}_N$ if $\|\pi_{\mathcal{K}}(x)\| \leq \|\pi_{\mathcal{H}}(x)\|$ for all $x \in M \otimes_{\mathrm{alg}} N^{\mathrm{op}}$.

As one expects, if $(\rho, \mathcal{K})$ and $(\pi, \mathcal{H})$ are unitary representations of Γ, then $\rho \prec \pi$ implies that $\mathcal{K}^\rho \prec \mathcal{H}^\pi$ (see, for instance, [AD93]).

We end this section by mentioning a fact that will be very useful in the sequel. If Γ is a countable group, then Γ is amenable if and only if the trivial representation is weakly contained in the left regular representation of Γ. Analogously, one can prove that if (M, τ) is a tracial von Neumann algebra, then M is amenable if and only if the trivial M-M-bimodule $\mathrm{L}^2(M)$ is weakly contained in the coarse M-M-bimodule $\mathrm{L}^2(M) \otimes \mathrm{L}^2(M)$.

2.3 Popa's intertwining-by-bimodules

As we have already remarked in the introduction, conjugating subalgebras of a von Neumann algebra by unitary elements is very important in the context of Popa's deformation/rigidity theory. The main result of this section is a criterion of Popa which helps us to decide whenever two subalgebras are (virtually) unitarily conjugate. To illustrate better what happens, let us start with a simple observation. Let (M, τ) be a tracial von Neumann algebra and let P and Q be two unital subalgebras that are unitarily conjugate, i.e. $u^*Pu = Q$, for some unitary $u \in \mathcal{U}(M)$. Then $\mathcal{H} := u\mathrm{L}^2(M)$ is a P-Q-subbimodule of ${}_P\mathrm{L}^2(M)_Q$, with $\dim(\mathcal{H}_Q) < \infty$ (more exactly, $\dim(\mathcal{H}_Q) = 1$). Such a bimodule is called an *intertwining bimodule* between P and Q. Ideally, we would like to prove also a converse statement, namely that the existence of an intertwining bimodule implies unitary conjugacy, but, in general, this is far from being true. But what Popa's criterion says is that the existence of an intertwining bimodule between P and Q forces a corner of P to be unitarily conjugate into a corner of Q. Moreover, in the special case when P and Q are Cartan subalgebras, this is sufficient to deduce unitary conjugacy.

Let (M, τ) be a tracial von Neumann algebra. Suppose that p and q are non-zero projections in M and that $P \subset pMp$ and $Q \subset qMq$ are von Neumann subalgebras.

We write $P \prec_M Q$ if there exists a non-zero P-Q-bimodule $\mathcal{H} \subset pL^2(M)q$ which has finite right Q-dimension. We write $P \prec_M^f Q$ if $Pp' \prec Q$, for all non-zero projections $p' \in P' \cap pMp$. If no confusion is possible, we simply write $P \prec Q$ and $P \prec^f Q$.

Theorem 2.3 ([Po03, Theorem 2.1 and Corollary 2.3])**.** *Let (M, τ) be a tracial von Neumann algebra. Assume that $p, q \in M$ are projections and that $P \subset pMp$ and $Q \subset qMq$ are von Neumann subalgebras with P being generated by a group of unitaries $\mathcal{G} \subset \mathcal{U}(P)$. Then the following three statements are equivalent.*

- *$P \prec_M Q$,*
- *There exist a non-zero partial isometry $v \in \mathrm{M}_{1,n}(\mathbb{C}) \otimes pMq$, a projection $q_0 \in \mathrm{M}_n(\mathbb{C}) \otimes Q$ and a normal $*$-homomorphism $\theta : P \to q_0(\mathrm{M}_n(\mathbb{C}) \otimes Q)q_0$ such that $xv = v\theta(x)$ for all $x \in P$.*

- *There is no sequence of unitaries* (w_n) *in* $\mathcal{G}$ *satisfying*
$$\|E_Q(x^* w_n y)\|_2 \to 0 \quad \textit{for all } x, y \in pMq \ .$$

If we assume that P and Q are Cartan subalgebras in a II_1 factor M, one gets a true conjugacy criterion.

Theorem 2.4 ([Po01, Theorem A.1])**.** *Let* (M, τ) *be a tracial von Neumann algebra and let* P *and* Q *be maximal abelian subalgebras. Then the following statements are equivalent.*

- $P \prec_M Q$*,*
- *There exists a non-zero partial isometry* $v \in M$ *such that* $vv^* \in P$*,* $v^*v \in Q$ *and* $v^*Pv = Qv^*v$*.*

Furthermore, if M *is a type* II_1 *factor and* P *and* Q *are Cartan subalgebras, a third statement is equivalent:*

- *There exists a unitary* $u \in \mathcal{U}(M)$ *such that* $uPu^* = Q$*.*

The next lemma is essentially a variant of Popa's criterion [Po01, Theorem A.1], but we give a complete proof.

Lemma 2.5. *Let* M *be a type* II_1 *factor and* $A \subset M$ *be a Cartan subalgebra. Let* $B \subset M$ *be an abelian subalgebra and* $\mathcal{G} < \mathcal{N}_M(B)$ *be a subgroup such that*

- $B' \cap M \prec A$*,*
- *the normalizer of* $B' \cap M$ *in* M *is a factor (or equivalently, the adjoint action of* $\mathcal{G}$ *on* $Z(B' \cap M)$ *is ergodic).*

Then there exist a projection $p \in A$ *and* $v \in \mathrm{M}_{1,n}(\mathbb{C}) \otimes Mp$ *such that* $vv^* = 1$*,* $v^*v = 1 \otimes p$ *and* $v^*(B' \cap M)v = \mathrm{M}_n(\mathbb{C}) \otimes Ap$*.*

Proof. Since $B' \cap M \prec A$, the von Neumann algebra $B' \cap M$ has a type I direct summand. Since the adjoint action of $\mathcal{G}$ on $Z(B' \cap M)$ is ergodic, we find an integer $n \geq 1$ such that $B' \cap M = \mathrm{M}_n(\mathbb{C}) \otimes Z(B' \cap M)$. So, we may take a system of matrix units $(e_{ij})_{1 \leq i,j \leq n}$ in $B' \cap M$ with $e := e_{11}$ satisfying $e(B' \cap M)e = Z(B' \cap M)e$. By construction, $Z(B' \cap M)e$ is a maximal abelian subalgebra of eMe, whose normalizer is a factor.

Since $B' \cap M \prec A$, also $Z(B' \cap M)e \prec A$ and hence, by [Po01, Theorem A.1], there exist a projection $p \in A$ and $v_0 \in \mathrm{M}_{n,1}(\mathbb{C}) \otimes Mp$ such that

$v_0 v_0^* = e$, $v_0^* v_0 = p$ and $v_0^*(B' \cap M)v_0 = Ap$. Define $v \in \mathrm{M}_{1,n}(\mathbb{C}) \otimes Mp$ by $v = \sum_{k=1}^{n} e_{1k} \otimes e_{1k} v_0$. Then one checks easily that $vv^* = 1$, $v^*v = 1 \otimes p$ and $v^*(B' \cap M)v = \mathrm{M}_n(\mathbb{C}) \otimes Ap$.

□

We leave the proof of the following easy lemma to the reader (see e.g. [Va10b, Proposition 2.6] for a special case).

Lemma 2.6. *Let (M, τ) be a tracial von Neumann algebra. Assume that $p, q \in M$ are projections and that $P \subset pMp$ and $Q \subset qMq$ are von Neumann subalgebras. Denote by $\mathcal{P} := \mathcal{N}_{pMp}(P)''$ the normalizer of P inside pMp. The set of projections*

$$\{p_0 \in P' \cap pMp \mid Pp_0 \prec^f Q\}$$

attains its maximum in a projection p_1 that belongs to $\mathcal{Z}(\mathcal{P})$. Moreover $P(p - p_1) \not\prec Q$.

Lemma 2.7 ([Va10b, Section 2])**.** *Let Γ be a countable group and $\Gamma \curvearrowright (B, \tau)$ a trace-preserving action. Put $M = B \rtimes \Gamma$. Let $p \in M$ be a projection and $P \subset pMp$ be a von Neumann subalgebra.*

(a) Assume that $\Lambda < \Gamma$ is a subgroup. The set of projections $p_0 \in P' \cap pMp$ satisfying $Pp_0 \prec^f B \rtimes \Lambda$ attains its maximum in a projection p_1 that belongs to the center of the normalizer of P inside pMp. Moreover $P(p - p_1) \not\prec B \rtimes \Lambda$.

(b) Assume that $\Lambda_1, \Lambda_2 < \Gamma$ are subgroups with $\Lambda_2 \lhd \Gamma$ being normal. If $P \prec^f B \rtimes \Lambda_j$ for all $j \in \{1, 2\}$, then $P \prec^f B \rtimes (\Lambda_1 \cap \Lambda_2)$.

Proof. The first statement follows from [Va10b, Proposition 2.6 and Lemma 2.5], while the second statement follows from [Va10b, Lemmas 2.7 and 2.5]. □

Lemma 2.8. *Let Γ be a countable group and $\Gamma \curvearrowright (B, \tau)$ a trace-preserving action. Put $M = B \rtimes \Gamma$ and let $p \in M$ be a projection. Assume that $Q \subset pMp$ is a von Neumann subalgebra that is normalized by a group of unitaries $\mathcal{G} \subset \mathcal{U}(pMp)$. Let $\Lambda < \Gamma$ be a subgroup.*

If $Q \prec^f B$ and $\mathcal{G}'' \prec B \rtimes \Lambda$, then $(Q \cup \mathcal{G})'' \prec B \rtimes \Lambda$.

Proof. For every subset $\mathcal{F} \subset \Gamma$, we denote by $P_{\mathcal{F}}$ the orthogonal projection of $L^2(M)$ onto the closed linear span of $\{bu_g \mid b \in B, g \in \mathcal{F}\}$. We say that a subset $\mathcal{F} \subset \Gamma$ is *small relative to* Λ if $\mathcal{F}$ is contained in a finite union of subsets of the form $g\Lambda h$ with $g, h \in \Gamma$.

Assume that $(Q \cup \mathcal{G})'' \not\prec B \rtimes \Lambda$. Since $\mathcal{U}(Q)\mathcal{G}$ is a group of unitaries generating $(Q \cup \mathcal{G})''$, we get from [Va10b, Lemma 2.4] two sequences of unitaries $a_n \in \mathcal{U}(Q)$ and $w_n \in \mathcal{G}$ such that $\|P_{\mathcal{F}}(a_n w_n)\|_2 \to 0$ for every subset $\mathcal{F} \subset \Gamma$ that is small relative to Λ.

Since $\mathcal{G}'' \prec B \rtimes \Lambda$, Theorem 2.3 provides a non-zero partial isometry $v \in \mathrm{M}_{1,n}(\mathbb{C}) \otimes pM$, a projection $q \in \mathrm{M}_n(\mathbb{C}) \otimes (B \rtimes \Lambda)$ and a normal $*$-homomorphism $\theta : \mathcal{G}'' \to q(\mathrm{M}_n(\mathbb{C}) \otimes (B \rtimes \Lambda))q$ such that $xv = v\theta(x)$ for all $x \in \mathcal{G}''$. Denote $p_1 := vv^*$ and fix $0 < \varepsilon < \|p_1\|_2/3$. By the Kaplansky density theorem, we can take a finite subset $\mathcal{F}_1 \subset \Gamma$ and an element v_1 in the linear span of $\{bu_g \mid b \in \mathrm{M}_{1,n}(\mathbb{C}) \otimes B, g \in \mathcal{F}_1\}$ such that $\|v_1\| \leq 1$ and $\|v - v_1\|_2 < \varepsilon$.

Denote $\mathcal{F}_2 := \mathcal{F}_1 \Lambda \mathcal{F}_1^{-1}$. Observe that $\mathcal{F}_2$ is small relative to Λ. Write $x_n := v_1 \theta(w_n) v_1^*$. By construction, every x_n lies in the image of $P_{\mathcal{F}_2}$ and we have that $\|x_n\| \leq 1$, $\|w_n p_1 - x_n\|_2 < 2\varepsilon$ for all n.

Since $Q \prec^f B$, we obtain from [Va10b, Lemma 2.5] a finite subset $\mathcal{F}_3 \subset \Gamma$ such that $\|a_n - P_{\mathcal{F}_3}(a_n)\|_2 < \varepsilon$ for all n. In combination with the previous paragraph, we get that $\|a_n w_n p_1 - P_{\mathcal{F}_3}(a_n) x_n\|_2 < 3\varepsilon$ for all n. Denote $\mathcal{F}_4 := \mathcal{F}_3 \mathcal{F}_2$ and observe that $\mathcal{F}_4$ is still small relative to Λ. By construction, $P_{\mathcal{F}_3}(a_n) x_n$ lies in the image of $P_{\mathcal{F}_4}$ and we have thus shown that $\|a_n w_n p_1 - P_{\mathcal{F}_4}(a_n w_n p_1)\|_2 < 3\varepsilon$ for all n.

Since $\|P_{\mathcal{F}}(a_n w_n)\|_2 \to 0$ for every subset $\mathcal{F} \subset \Gamma$ that is small relative to Λ, it follows from [Va10b, Lemma 2.3] that $\|P_{\mathcal{F}_4}(a_n w_n p_1)\|_2 \to 0$. Hence $\limsup_n \|a_n w_n p_1\|_2 \leq 3\varepsilon$. Since a_n and w_n are unitaries, we arrive at the contradiction that $\|p_1\|_2 \leq 3\varepsilon < \|p_1\|_2$. $\square$

2.4 Relative amenability

Let (M, τ) be a tracial von Neumann algebra and $Q \subset M$ be a von Neumann subalgebra. Recall that the Jones basic construction $\langle M, e_Q \rangle$ is defined as the von Neumann algebra acting on $L^2(M)$ generated by M and the orthogonal projection e_Q of $L^2(M)$ onto $L^2(Q)$. Moreover, we have that $\langle M, e_Q \rangle$ equals the commutant of the right Q-action on $L^2(M)$, i.e. $\langle M, e_Q \rangle = B(L^2(M)) \cap (Q^{\mathrm{op}})'$.

Recall that a tracial von Neumann algebra (M, τ) is amenable if there exists an M-central state on $B(L^2(M))$ whose restriction to M equals τ. As we noticed before, M is amenable if and only if the trivial M-M-bimodule $\mathrm{L}^2(M)$ is weakly contained in the coarse M-M-bimodule $\mathrm{L}^2(M) \otimes \mathrm{L}^2(M)$.

Definition 2.9 ([OP07, Section 2.2])**.** Let (M, τ) be a tracial von Neumann algebra and let $P \subset pMp$ and $Q \subset M$ be von Neumann subalgebras. We say that P is *amenable relative to* Q inside M, if there exists a P-central positive functional on the von Neumann algebra $p\langle M, e_Q\rangle p$ whose restriction to pMp equals τ.

Following [IPV10], we say that P is *strongly non-amenable relative to* Q inside M if, for all non-zero projections $q \in P' \cap pMp$, we have that Pq is non-amenable relative to Q inside M.

Theorem 2.10 ([OP07, Theorem 2.1])**.** *Let (M, τ) be a tracial von Neumann algebra and let $P, Q \subset M$ be von Neumann subalgebras. The following statements are equivalent.*

- *P is amenable relative to Q inside M,*
- *${}_P L^2(M)_M$ is weakly contained in ${}_P L^2(\langle M, e_Q\rangle)_M$,*
- *There exists a P-central state φ on $\langle M, e_Q\rangle$ such that the restriction of φ to M is normal and the restriction of φ to the center of $P' \cap M$ is faithful,*
- *There exists a conditional expectation $\Phi : \langle M, e_Q\rangle \to P$ such that the restriction of Φ to M equals the expectation $E_P : M \to P$,*
- *There exists a net $(\xi_i)_{i\in I} \in L^2(\langle M, e_Q\rangle)$ such that*

$$\lim_{i\in I} \langle x\xi_i, \xi_i\rangle = \tau(x), \forall x \in M \text{ and } \lim_{i\in I} \|a\xi_i - \xi_i a\|_2 = 0, \forall a \in P.$$

Lemma 2.11. *Let (M, τ) be a tracial von Neumann algebra and assume that $M \subset \widetilde{M}$, for some von Neumann algebra $\widetilde{M}$. Let $S \subset M$ be a subset and let Ω be a positive functional on $\widetilde{M}$ such that the restriction of Ω to M is bounded by $c\tau$, for some constant $c > 0$. If Ω is S-central, then Ω is S''-central.*

Proof. For all $y \in \widetilde{M}$ and $x \in M$, by the Cauchy-Schwarz inequality, we have that $|\Omega(yx)|^2 \leq \Omega(y^*y)\Omega(x^*x) \leq c\Omega(y^*y)\tau(x^*x) \leq c\,\|y\|^2 \cdot \|x\|_2^2$ and similarly $|\Omega(xy)|^2 \leq c\,\|y\|^2 \cdot \|x\|_2^2$.

Thus, the set $M_0 := \{x \in M \mid \Omega(xy) = \Omega(yx) \text{ for all } y \in \widetilde{M}\}$ is an L^2-closed $*$-subalgebra of M. Since S is contained in M_0 and M_0 is L^2-closed, it follows that S'' is also contained in M_0, and this exactly means that Ω is S''-central. $\square$

Similarly, if Γ is a countable group with subgroups $\Lambda_1, \Lambda_2 < \Gamma$, we say that Λ_1 is amenable relative to Λ_2 if the action of Λ_1 on Γ/Λ_2 by left translations admits an invariant mean. If $\Lambda < \Gamma$ is a subgroup, we say that Λ is co-amenable in Γ if Γ is amenable relative to Λ.

The following lemma is essentially contained in [MP03, Proposition 6]. For completeness, we provide a full proof.

Lemma 2.12. *Let Γ be a countable group and $\Gamma \curvearrowright (B,\tau)$ a trace-preserving action. Put $M = B \rtimes \Gamma$ and let $\Lambda_1, \Lambda_2 < \Gamma$ be subgroups. Then the following statements are equivalent.*

(a) $B \rtimes \Lambda_1$ is amenable relative to $B \rtimes \Lambda_2$ inside M,

(b) $L\Lambda_1$ is amenable relative to $B \rtimes \Lambda_2$ inside M,

(c) Λ_1 is amenable relative to Λ_2 inside Γ.

Proof. (a) $\Rightarrow$ (b) is trivial.

(b) $\Rightarrow$ (c). For every $g \in \Gamma$, we denote by $\delta_{g\Lambda_2} \in \ell^\infty(\Gamma/\Lambda_2)$ the function that is equal to 1 in $g\Lambda_2$ and that is equal to 0 elsewhere. By Lemma 2.2, there is a unique unital normal $*$-homomorphism

$$\pi : \ell^\infty(\Gamma/\Lambda_2) \to \langle M, e_{B\rtimes\Lambda_2}\rangle$$

satisfying $\pi(\delta_{g\Lambda_2}) = u_g \, e_{B\rtimes\Lambda_2} \, u_g^*$, for all $g \in \Gamma$. By construction, π conjugates the left translation action of Γ on $\ell^\infty(\Gamma/\Lambda_2)$ with the action $(\mathrm{Ad}\, u_g)_{g\in\Gamma}$. Since $L\Lambda_1$ is amenable relative to $B \rtimes \Lambda_2$ inside M, we can take an $L\Lambda_1$-central state Ω on $\langle M, e_{B\rtimes\Lambda_2}\rangle$. Then $\Omega \circ \pi$ is a Λ_1-invariant state on $\ell^\infty(\Gamma/\Lambda_2)$. Hence (c) holds.

(c) $\Rightarrow$ (a). We denote by $\eta : \Gamma \to \mathcal{U}(\ell^2(\Gamma/\Lambda_2))$ the unitary representation of Γ given by left translation operators. We then turn the Hilbert space $L^2(M) \otimes \ell^2(\Gamma/\Lambda_2)$ into an M-M-bimodule with the bimodule action given by

$$(bu_g) \cdot (x \otimes \xi) \cdot y := bu_g xy \otimes \eta_g \xi \quad \text{for all } \; b \in B, g \in \Gamma, x, y \in M, \xi \in \ell^2(\Gamma/\Lambda_2) \, .$$

Since (c) holds, take a sequence of unit vectors $\xi_n \in \ell^2(\Gamma/\Lambda_2)$ satisfying $\lim_n \|\eta_g \xi_n - \xi_n\|_2 = 0$ for all $g \in \Lambda_1$. Then the sequence of vectors $1 \otimes \xi_n \in L^2(M) \otimes \ell^2(\Gamma/\Lambda_2)$ satisfies

$$\langle x \cdot (1 \otimes \xi_n), 1 \otimes \xi_n \rangle = \tau(x), \text{ for all } x \in M,$$

and

$$\lim_n \|bu_g \cdot (1 \otimes \xi_n) - (1 \otimes \xi_n) \cdot bu_g\|_2 = 0,$$

for all $b \in \mathcal{U}(B),\ g \in \Lambda_1$. By Lemma 2.2, there is a unique unitary operator

$$\theta : L^2(\langle M, e_{B\rtimes\Lambda_2}\rangle) \to L^2(M) \otimes \ell^2(\Gamma/\Lambda_2)$$

satisfying $\theta(bu_g\, e_{B\rtimes\Lambda_2}\, x) = bu_g x \otimes \delta_{g\Lambda_2}$, for all $b \in B,\ g \in \Gamma,\ x \in M$. This unitary θ is M-M-bimodular. Define $S_n \in L^2(\langle M, e_{B\rtimes\Lambda_2}\rangle)$ given by $S_n := \theta^{-1}(1 \otimes \xi_n)$. Choose a state Ω on $\langle M, e_{B\rtimes\Lambda_2}\rangle$ as a weak*-limit point of the sequence of states $T \mapsto \langle TS_n, S_n\rangle$. By construction, $\Omega(x) = \tau(x)$ for all $x \in M$ and Ω is $\mathcal{G}$-central, where $\mathcal{G} = \{bu_g \mid b \in \mathcal{U}(B), g \in \Lambda_1\}$. Using Lemma 2.11, it follows that Ω is $(B \rtimes \Lambda_1)$-central. So (a) holds. □

Lemma 2.13. *Let (M, τ) be a tracial von Neumann algebra and let $P \subset pMp$ and $Q \subset M$ be von Neumann subalgebras. The set of projections $p_0 \in P' \cap pMp$ with the property that Pp_0 is amenable relative to Q, attains its maximum in a projection p_1 that belongs to the center of the normalizer of P inside pMp.*

Proof. Denote by $\mathcal{P}$ the set of projections $p_0 \in P' \cap pMp$ with the property that Pp_0 is amenable relative to Q. If $p_0 \in \mathcal{P}$ and $u \in \mathcal{N}_{pMp}(P)$, it is easy to check that $up_0u^* \in \mathcal{P}$. It therefore suffices to prove the following two statements.

1. *I*f $p_0, p_1 \in \mathcal{P}$, then $q := p_0 \vee p_1$ belongs to $\mathcal{P}$. For all $j \in \{0, 1\}$, choose Pp_j-central positive functionals Ω_j on $p_j\langle M, e_Q\rangle p_j$ with the property that $\Omega_j(x) = \tau(x)$ for all $x \in p_jMp_j$. Define the positive functional Ω on $q\langle M, e_Q\rangle q$ by the formula $\Omega(T) := \Omega_0(p_0Tp_0) + \Omega_1(p_1Tp_1)$. It is easy to check that Ω is Pq-central and that the restriction of Ω to qMq is normal and faithful. By Theorem 2.10, we get that Pq is amenable relative to Q.

2. *I*f p_n is an increasing sequence in $\mathcal{P}$ that converges strongly to q, then also $q \in \mathcal{P}$. Take Pp_n-central positive functionals Ω_n on $p_n\langle M, e_Q\rangle p_n$ with the property that $\Omega_n(x) = \tau(x)$ for all $n \in \mathbb{N}$ and all $x \in p_nMp_n$. Choose a positive functional Ω on $q\langle M, e_Q\rangle q$ as a weak* limit point of the sequence of functionals $T \mapsto \Omega_n(p_nTp_n)$. By construction, Ω is Pq-central and $\Omega(x) = \tau(x)$ for all $x \in qMq$. So $q \in \mathcal{P}$. □

We also need the following special case of [PV11, Proposition 2.7].

Lemma 2.14 ([PV11, Proposition 2.7])**.** *Let Γ be a countable group and $\Gamma \curvearrowright (B, \tau)$ a trace-preserving action. Put $M = B \rtimes \Gamma$. Let $p \in M$ be a projection and $P \subset pMp$ a von Neumann subalgebra. Assume that $\Lambda_1, \Lambda_2 < \Gamma$ are subgroups with $\Lambda_2 \lhd \Gamma$ being normal. If P is amenable relative to $B \rtimes \Lambda_j$ for all $j \in \{1, 2\}$, then P is amenable relative to $B \rtimes (\Lambda_1 \cap \Lambda_2)$.*

Lemma 2.15. *Let $\sigma : \Gamma \curvearrowright (X, \mu)$ be a free p.m.p. action of a countable group Γ on a standard probability space (X, μ). Denote $A := L^\infty(X, \mu)$ and let $p \in A$*

be a non-zero projection. Let $\Sigma < \Gamma$ be a subgroup, $n \geq 1$ be an integer and denote $M := \mathrm{M}_n(\mathbb{C}) \otimes p(A \rtimes \Gamma)p$ and $Q := \mathrm{M}_n(\mathbb{C}) \otimes p(A \rtimes \Sigma)p$. Assume that $\mathcal{G} < \mathcal{U}(M)$ is a subgroup and $q \in \mathcal{G}' \cap M$ is a non-zero projection such that

- *$\mathcal{G}$ normalizes $\mathrm{M}_n(\mathbb{C}) \otimes Ap$,*
- *$(\mathcal{G}q)''$ is amenable relative to Q.*

Denote by M_0 the von Neumann algebra generated by $\mathcal{G}$ and $1 \otimes Ap$. Then there exists a non-zero projection $q_0 \in M_0' \cap M$ such that $M_0 q_0$ is amenable relative to Q.

Proof. Since $(\mathcal{G}q)''$ is amenable relative to Q, there exists a state Ω_1 on $q\langle M, e_Q\rangle q$ such that Ω_1 is $\mathcal{G}q$-central and it restricts to the trace on qMq.

Denote $N := \mathrm{M}_n(\mathbb{C}) \otimes (p \otimes 1)((A \mathbin{\overline{\otimes}} \ell^\infty(\Gamma/\Sigma)) \rtimes \Gamma)(p \otimes 1)$, where $\sigma : \Gamma \curvearrowright A \mathbin{\overline{\otimes}} \ell^\infty(\Gamma/\Sigma)$ is the diagonal action. By Lemma 2.2 it follows that N is isomorphic with the basic construction $\langle M, e_Q\rangle$, thus Ω_1 is a $\mathcal{G}q$-central state on qNq whose restriction to qMq equals the trace.

Define a state Ω on N by the formula $\Omega(T) = \Omega_1(qTq)$, for all $T \in N$. Since q commutes with $\mathcal{G}$, it follows immediately that Ω is $\mathcal{G}$-central. Since Ω_1 restricts to the trace on qMq, we get that $\Omega \mid_M$ is bounded by a multiple of the trace.

Denote $D := \mathrm{M}_n(\mathbb{C}) \mathbin{\overline{\otimes}} Ap \mathbin{\overline{\otimes}} \ell^\infty(\Gamma/\Sigma)$ and let $E_D : N \to D$ be the unique trace-preserving conditional expectation.

We claim that every unitary $v \in \mathcal{U}(M)$ that normalizes $\mathrm{M}_n(\mathbb{C}) \otimes Ap$ also normalizes D inside N. Indeed, take $v \in \mathcal{N}_M(\mathrm{M}_n(\mathbb{C}) \otimes Ap)$. For every $g \in \Gamma$, there exist a projection $p_g \in Ap$ and a unitary $v_g \in \mathcal{U}(\mathrm{M}_n(\mathbb{C}) \otimes A\sigma_g(p_g))$ such that $\sum_{g\in\Gamma} p_g = 1$, $\sum_{g\in\Gamma} \sigma_g(p_g) = 1$ and $v(1 \otimes p_g) = v_g(1 \otimes u_g)$. If $x \in \mathrm{M}_n(\mathbb{C}) \otimes Ap_g$, then it follows immediately that $vxv^* = v_g(\mathrm{id} \otimes \sigma_g)(x)v_g^*$. Moreover, for every element $x \in \mathrm{M}_n(\mathbb{C}) \mathbin{\overline{\otimes}} Ap_g \mathbin{\overline{\otimes}} \ell^\infty(\Gamma/\Sigma)$, we get that $vxv^* = (v_g \otimes 1)(\mathrm{id} \otimes \sigma_g)(x)(v_g^* \otimes 1)$, and since the right hand side belongs to D, our claim is proven.

If $v \in \mathcal{N}_M(\mathrm{M}_n(\mathbb{C}) \otimes Ap)$, then $v \in \mathcal{N}_N(D)$ and since E_D is the unique trace-preserving conditional expectation form N onto D, it follows that $E_D(vTv^*) = vE_D(T)v^*$, for all $T \in N$. In particular, $E_D(vTv^*) = vE_D(T)v^*$, for all $v \in \mathcal{G}$ and $T \in N$.

Define a state $\widetilde{\Omega}$ on N by $\widetilde{\Omega}(T) = \Omega(E_D(T))$, for all $T \in N$. Since Ω is $\mathcal{G}$-central, the previous remark implies that $\widetilde{\Omega}$ is also $\mathcal{G}$-central. Since $E_D(M) \subset \mathrm{M}_n(\mathbb{C}) \otimes Ap$, we have that $\widetilde{\Omega} \mid_M$ is bounded by a multiple of the trace. Notice that $\widetilde{\Omega}$ is automatically $(1 \otimes Ap)$-central, since $1 \otimes Ap$ commutes with D, and hence, by Lemma 2.11, it follows that $\widetilde{\Omega}$ is an M_0-central state on N whose restriction to M is bounded by a multiple of the trace. In particular, $\widetilde{\Omega}$ is normal on M, and then, by [BV12, Lemma 2.9], there exists a non-zero projection $q_0 \in M_0' \cap M$ such that $M_0 q_0$ is amenable relative to Q. $\square$

Let (M, τ) be a tracial von Neumann algebra and $Q \subset M$ be a von Neumann algebra. Recall that the basic construction $\langle M, e_Q \rangle$ is defined as $\langle M, e_Q \rangle = B(L^2(M)) \cap (Q^{\mathrm{op}})'$. Replacing in Definition 2.9 the trivial M-M-bimodule $\mathrm{L}^2(M)$ by an arbitrary M-M-bimodule $\mathcal{H}$, we arrive at the following definition (cf. [Si10, Theorem 2.2] and [PV11, Definition 2.3]).

Definition 2.16. Let (M, τ) and (N, τ) be tracial von Neumann algebras. Let $P \subset M$ be a von Neumann subalgebra. An M-N-bimodule ${}_M\mathcal{H}_N$ is said to be *left P-amenable* if $B(\mathcal{H}) \cap (N^{\mathrm{op}})'$ admits a P-central state whose restriction to M equals τ.

If (M, τ) is a tracial von Neumann algebra and if $P \subset pMp$, $Q \subset M$ are von Neumann subalgebras, then by definition, P is amenable relative to Q if and only if the pMp-Q-bimodule $pL^2(M)$ is left P-amenable. Even more specifically, recall from [Po86, Definition 3.2.1] and [AD93, Definition 2.1] that $Q \subset M$ is called *co-amenable* if M is amenable relative to Q. Therefore, Q is co-amenable in M if and only if the M-Q-bimodule ${}_M\mathrm{L}^2(M)_Q$ is left M-amenable.

The notion of amenability for bimodules, first introduced in [AD93], has its origins in the concept of amenable representation of [Be89]. To make this link more precise, assume that $\Gamma \curvearrowright (B, \tau)$ is a trace-preserving action of a countable group Γ and put $M := B \rtimes \Gamma$. If $(\pi, \mathcal{H})$ is a unitary representation of Γ, we define the M-M-bimodule $\mathcal{H}^\pi$. Then the M-M-bimodule $\mathcal{H}^\pi$ is left M-amenable if and only if π is an amenable representation in the sense of [Be89, Definition 1.1], i.e. if and only if $B(\mathcal{H})$ admits an $\mathrm{Ad}(\pi_g)_{g \in \Gamma}$-invariant state (cf. [AD93, Proposition 3.3]).

The following easy lemmas are essentially contained in [OP07, Section 2.2]. For completeness, we provide full proofs.

Lemma 2.17. *Let (M, τ) and (N, τ) be tracial von Neumann algebras. Let $P \subset M$ be a von Neumann subalgebra and ${}_M\mathcal{K}_N$ an M-N-bimodule. The following two statements are equivalent.*

(a) There exists a non-zero P-central positive functional on $B(\mathcal{K}) \cap (N^{op})'$ whose restriction to M is normal,

(b) There exists a non-zero projection $p \in P' \cap M$ such that the pMp-N-bimodule ${}_{pMp}(p\mathcal{K})_N$ is left Pp-amenable.

Proof. (a) $\Rightarrow$ (b). Let Ω be a non-zero P-central positive functional on $\mathcal{N} := B(\mathcal{K}) \cap (N^{\mathrm{op}})'$ whose restriction to M, denoted by ω is normal. Take $T \in L^1(M)^+$ such that $\omega(x) = \tau(xT)$ for all $x \in M$. Note that $T \neq 0$. Since ω is P-central, we have that $T \in L^1(P' \cap M)$. Take $\varepsilon > 0$ small enough such that the spectral projection $p := \chi_{(\varepsilon,+\infty)}(T)$ is non-zero. Note that $p \in P' \cap M$ and that we can take $S \in p(P' \cap M)^+ p$ such that $TS = ST = p$. The formula $y \mapsto \Omega(S^{1/2} y S^{1/2})$ defines Pp-central positive functional on $B(p\mathcal{K}) \cap (N^{\mathrm{op}})'$ whose restriction to pMp equals τ. So ${}_{pMp}(p\mathcal{K})_N$ is left Pp-amenable.

(b) $\Rightarrow$ (a). Assume that $p \in P' \cap M$ is a non-zero projection and that Ω is a Pp-central positive functional on $B(p\mathcal{K}) \cap (N^{\mathrm{op}})'$ whose restriction to pMp equals τ. Then the formula $y \mapsto \Omega(pyp)$ defines a non-zero P-central positive functional on $B(\mathcal{K}) \cap (N^{\mathrm{op}})'$ whose restriction to M is normal. □

Lemma 2.18. *Let (M, τ) be a tracial von Neumann algebra and let $P \subset M$ be a von Neumann subalgebra. Let $\mathcal{K}$ be an M-M-bimodule. Assume that $\xi_n \in \mathcal{K}$ is a sequence of vectors and $\varepsilon > 0$ such that*

- $\|x\xi_n\| \leq \|x\|_2$ *for all $x \in M$ and $n \in \mathbb{N}$,*
- $\|\xi_n\| \geq \varepsilon$ *for all $n \in \mathbb{N}$,*
- *for all $x \in P$, we have that* $\lim_n \|x\xi_n - \xi_n x\| = 0$.

Then there exists a non-zero projection $p \in P' \cap M$ such that the pMp-M-bimodule $p\mathcal{K}$ is left Pp-amenable.

Proof. Choose a positive functional Ω on $B(\mathcal{K}) \cap (M^{\mathrm{op}})'$ as a weak* limit point of the sequence of positive functionals $y \mapsto \langle y\xi_n, \xi_n \rangle$. The conditions on ξ_n imply that $\Omega(x) \leq \tau(x)$ for all $x \in M^+$, that $\Omega(1) \geq \varepsilon^2$ and that Ω is P-central. In particular, Ω is non-zero and the restriction of Ω to M is normal. The conclusion now follows from Lemma 2.17. □

Lemma 2.19. *Let (M, τ) and (N, τ) be tracial von Neumann algebras. Let $P \subset M$ be a von Neumann subalgebra. Assume that for all $j \in \{1, \ldots, \ell\}$, we are given an M-N-bimodule $\mathcal{K}_j$. If $\bigoplus_{j=1}^{\ell} \mathcal{K}_j$ is a left P-amenable M-N-bimodule, then there exists a $j \in \{1, \ldots, \ell\}$ and a non-zero projection $p \in P' \cap M$ such that $p\mathcal{K}_j$ is a left Pp-amenable pMp-N-bimodule.*

Proof. Put $\mathcal{K} := \bigoplus_{j=1}^{\ell} \mathcal{K}_j$ and denote by p_j the orthogonal projection of $\mathcal{K}$ onto $\mathcal{K}_j$. Let Ω be a P-central state on $B(\mathcal{K}) \cap (N^{\mathrm{op}})'$ whose restriction to M equals τ. Take $j \in \{1, \ldots, \ell\}$ such that $\Omega(p_j) \neq 0$. Then the formula $y \mapsto \Omega(p_j y p_j)$ defines a non-zero P-central positive functional on $B(\mathcal{K}_j) \cap (N^{\mathrm{op}})'$ whose restriction to

M is smaller or equal than τ and hence normal. So the conclusion follows from Lemma 2.17. $\square$

2.5 Weak amenability and class $\mathcal{S}$

In this section, we briefly introduce weak amenability and Ozawa's class $\mathcal{S}$. We only use these concepts in the following way: the first two families of groups in Theorem 1.2 are weakly amenable and in class $\mathcal{S}$, so that we can apply the results of [PV12] to them. For more details and precise references, see for example [BO08].

Let A and B be C*-algebras and $\varphi : A \to B$ be a linear map. We say that φ is *completely bounded* (abbreviated c.b.) if

$$\|\varphi\|_{\text{cb}} := \sup_{n\in\mathbb{N}} \|\varphi_n : \mathbb{M}_n(A) \to \mathbb{M}_n(B)\| < \infty,$$

where φ_n is defined by $\varphi_n([a_{i,j}]) = [\varphi(a_{i,j})]$, for every matrix $[a_{i,j}] \in \mathbb{M}_n(A)$.

Let Γ be a countable group. A function $\varphi : \Gamma \to \mathbb{C}$ is called a *Herz-Schur multiplier* if the map $u_g \mapsto \varphi(g)u_g$ extends to an ultraweakly continuous, completely bounded linear map $m_\varphi : \mathrm{L}\Gamma \to \mathrm{L}\Gamma$. Whenever φ is a Herz-Schur multiplier, we denote $\|\varphi\|_{\text{cb}} := \|m_\varphi\|_{\text{cb}}$.

If Γ is a countable group, then an *approximate identity* on Γ is a sequence of finitely supported functions $\varphi_n : \Gamma \to \mathbb{C}$ such that $\varphi_n \to 1$ pointwise. Recall that the group Γ is amenable if and only if it admits an approximate identity consisting of positive definite functions.

In the same spirit, Cowling and Haagerup introduced in [CH88] the notion of weak amenability. A countable group Γ is called *weakly amenable* if it admits an approximate identity $\varphi_n : \Gamma \to \mathbb{C}$ such that $\sup_{n\in\mathbb{N}} \|\varphi_n\|_{\text{cb}} < \infty$. Amenable groups are clearly weakly amenable since for any positive definite function $\varphi : \Gamma \to \mathbb{C}$ we have that $\|\varphi\|_{\text{cb}} = \varphi(1)$. By [Ha78], [Co82], [dCH85], [CH88], the free groups and the groups in Theorem 1.2.(b) are weakly amenable. Ozawa [Oz07] proved that hyperbolic groups are also weakly amenable.

A countable group Γ is called *exact* if its reduced group C*-algebra is an exact C*-algebra, or equivalently, if Γ admits a topologically amenable action on a compact space. To make this more precise, assume that Γ acts by homeomorphisms on a compact space X. The action $\Gamma \curvearrowright X$ is said to be *topologically amenable* if there exists a sequence of continuous maps $\varphi_n : X \to \mathrm{Prob}(\Gamma)$ such that

$$\lim_{n\to\infty} \sup_{x\in X} \|g.\varphi_n(x) - \varphi_n(g.x)\|_1 = 0, \text{ for all } g \in \Gamma.$$

Here, $\mathrm{Prob}(\Gamma)$ denotes the set of all probability measures on Γ, seen as a subset of $\ell^1(\Gamma)$, i.e. $\mathrm{Prob}(\Gamma) = \{\mu \in \ell^1(\Gamma) \mid \mu \geq 0,\ \sum_{g\in\Gamma} \mu(g) = 1\}$. Notice that Γ acts naturally on $\mathrm{Prob}(\Gamma)$ by left translation: $g.\mu(h) = \mu(g^{-1}h)$, for all $\mu \in \mathrm{Prob}(\Gamma)$ and $g, h \in \Gamma$.

A first example of an exact group is the free group $\mathbb{F}_n$ with $n \geq 2$, whose left translation action on the boundary $\partial\mathbb{F}_n$ is topologically amenable. More generally, any hyperbolic group Γ is exact since the action of Γ on its Gromov compactification $\overline{\Gamma} := \Gamma \cup \partial\Gamma$ is topologically amenable by [Ad94]. Moreover one has that the right translation action of Γ on itself extends to a continuous action on the compactification $\overline{\Gamma}$ which is trivial on the hyperbolic boundary $\partial\Gamma$. Thus, in the case of a hyperbolic group we have certain information about both the left and the right translation actions on the Gromov compactification. This fact leads us to the notion of bi-exactness.

Following [Oz03], a group Γ is said to be *bi-exact* or in *class* $\mathcal{S}$ if Γ is an exact group and if there exists a map $\mu : \Gamma \to \mathrm{Prob}(\Gamma)$ satisfying

$$\lim_{k\to\infty} \|\mu(gkh) - g.\mu(k)\|_1 = 0 \quad \text{for all } \ g, h \in \Gamma\ .$$

It immediately follows that if Γ belongs to class $\mathcal{S}$ and if $\Lambda < \Gamma$ is an infinite subgroup, then the centralizer of Λ inside Γ is amenable. Ozawa's theorem in [Oz03] says that much more is true: if $Q \subset L\Gamma$ is any diffuse von Neumann subalgebra, then the relative commutant $Q' \cap L\Gamma$ is amenable. A type II_1 factor having this property is called *solid*. In particular, if Γ is any i.c.c. hyperbolic group, then $\mathrm{L}\Gamma$ is solid.

2.6 Weakly mixing actions

Let Γ be a countable group and let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation. The representation π is said to be *mixing* if, for all $\xi, \eta \in \mathcal{K}$, we have that $\langle \pi(g)\xi, \eta\rangle \to 0$, as $g \to \infty$, and *weakly mixing* if π has no non-zero finite-dimensional globally $(\pi(g))_{g\in\Gamma}$-invariant subspaces.

Similarly, a probability measure preserving action $\Gamma \curvearrowright (X, \mu)$ is called *weakly mixing* if the associated unitary representation $\Gamma \curvearrowright L^2(X) \ominus \mathbb{C}1$ is weakly mixing. If $\Gamma \curvearrowright (X, \mu)$ is a p.m.p. action, then the following conditions are equivalent:

- $\Gamma \curvearrowright (X, \mu)$ is weakly mixing,
- the diagonal action $\Gamma \curvearrowright X \times X : g \cdot (x, y) = (g \cdot x, g \cdot y)$ is ergodic,

- whenever $\Gamma \curvearrowright (Y,\eta)$ is a p.m.p. action and $F : X \times Y \to \mathbb{C}$ is a measurable function that is invariant under the diagonal action $\Gamma \curvearrowright X \times Y : g \cdot (x,y) = (g \cdot x, g \cdot y)$, we have that F is a.e. equal to a function that only depends on the Y-variable.

The following lemma is classical (see e.g. [PV06, Proposition 2.3 and Lemma 2.4] for a simple proof).

Lemma 2.20. *Assume that the countable group Γ acts on the countable set I. Let (X_0, μ_0) be an arbitrary non-trivial standard probability space. Then the following conditions are equivalent.*

- *For every $i \in I$, the orbit $\Gamma \cdot i$ is infinite,*
- *For every finite subset $\mathcal{F} \subset I$, there exists a $g \in \Gamma$ such that $g \cdot \mathcal{F} \cap \mathcal{F} = \emptyset$,*
- *The unitary representation $\Gamma \curvearrowright \ell^2(I)$ is weakly mixing,*
- *The generalized Bernoulli action $\Gamma \curvearrowright (X_0, \mu_0)^I$ is weakly mixing.*

2.7 Properties of amplified comultiplications

If (M_0, τ) is a II_1 factor, then the von Neumann algebra $M_n(\mathbb{C}) \otimes M_0$ is still a II_1 factor and this allows us to define, for any $r > 0$, the *amplification* M_0^r of M_0 as $M_0^r = p(M_n(\mathbb{C}) \otimes M_0)p$, where p is a projection in $M_n(\mathbb{C}) \otimes M_0$ with $(\mathrm{Tr} \otimes \tau)(p) = r$. Whenever M_0 is a II_1 factor and (M, τ) is a tracial von Neumann algebra such that $M_0 \subset M$, we define as follows the inclusion $M_0^r \subset M^r$. Choose a projection $p \in \mathrm{M}_n(\mathbb{C}) \otimes M_0$ with $(\mathrm{Tr} \otimes \tau)(p) = r$ and define $M_0^r = p(\mathrm{M}_n(\mathbb{C}) \otimes M_0)p$ and $M^r = p(\mathrm{M}_n(\mathbb{C}) \otimes M)p$. As such, the inclusion $M_0^r \subset M^r$ is defined up to conjugacy by a partial isometry in $\mathrm{M}_n(\mathbb{C}) \otimes M_0$.

Throughout this section, assume that M_0 is a II_1 factor and $r > 0$ such that $M_0^r = \mathrm{L}\Lambda$ for some countable group Λ. We denote by $(v_s)_{s \in \Lambda}$ the canonical generating unitaries of $\mathrm{L}\Lambda$ and define the *comultiplication* $\Delta : \mathrm{L}\Lambda \to \mathrm{L}\Lambda \mathbin{\overline{\otimes}} \mathrm{L}\Lambda$ given by $\Delta(v_s) = v_s \otimes v_s$ for all $s \in \Lambda$. Up to unitary conjugacy, we have a uniquely defined *amplified comultiplication* $\Delta : M_0 \to (M_0 \mathbin{\overline{\otimes}} M_0)^r$ that we continue to denote by Δ.

At a certain point, we will need the explicit relation between the original comultiplication on $\mathrm{L}\Lambda$ and the amplified comultiplication on M_0. This is spelt out in Remark 2.22.

Apart from statement (d), the following result is essentially contained in [IPV10, Proposition 7.2]. For completeness, we nevertheless give a full proof. At a first reading of Proposition 2.21, one may very well assume that $M_0 = M$, which is sufficient to prove Theorem 1.2.1. The most general setup is only needed to prove Theorem 1.2.2.

Proposition 2.21. *Let M_0 be a II_1 factor and $r > 0$ such that $M_0^r = L\Lambda$ for some countable group Λ. As above, denote by $\Delta : M_0 \to (M_0 \mathbin{\overline{\otimes}} M_0)^r$ the amplified comultiplication. Assume that M and $\widetilde{M}$ are tracial von Neumann algebras such that $M_0 \subset M$ and $M_0 \subset \widetilde{M}$.*

- *(a) We have $\Delta(M)' \cap (M \mathbin{\overline{\otimes}} M)^r = \mathbb{C}1$.*
- *(b) If $A \subset M$ is a diffuse von Neumann subalgebra, we have that $\Delta(A) \not\prec M \otimes 1$ and $\Delta(A) \not\prec 1 \otimes M$.*
- *(c) If $P \subset M$ is a von Neumann subalgebra and $M_0 \not\prec_M P$, then $\Delta(M_0) \not\prec_{M^r \overline{\otimes} M} M^r \mathbin{\overline{\otimes}} P$.*
- *(d) If $P \subset \widetilde{M}$ is a von Neumann subalgebra and $\Delta(M_0)$ is amenable relative to $M^r \mathbin{\overline{\otimes}} P$ inside $M^r \mathbin{\overline{\otimes}} \widetilde{M}$, then M_0 is amenable relative to P inside $\widetilde{M}$.*
- *(e) If $P \subset M_0$ is a von Neumann subalgebra that has no amenable direct summand, then we have that $\Delta(P)$ is strongly non-amenable relative to $M^r \otimes 1$.*

Proof. Throughout the proof, we fix a projection $p \in \mathrm{M}_n(\mathbb{C}) \otimes M_0$ with $(\mathrm{Tr} \otimes \tau)(p) = r$. We identify $p(\mathrm{M}_n(\mathbb{C}) \otimes M_0)p = \mathrm{L}\Lambda$.

(c) Let $\Delta : \mathrm{L}\Lambda \to \mathrm{L}\Lambda \mathbin{\overline{\otimes}} \mathrm{L}\Lambda : \Delta(v_s) = v_s \otimes v_s$ be the original comultiplication. Since $M_0 \not\prec_M P$, also $M_0^r \not\prec_M P$. By Theorem 2.3, we can take a sequence $s_n \in \Lambda$ such that

$$\|E_P(x^* v_{s_n} y)\|_2 \to 0 \quad \text{for all } \ x, y \in p(\mathbb{C}^n \otimes M) \ .$$

We claim that

$$\|E_{M \overline{\otimes} P}(x^* \Delta(v_{s_n}) y)\|_2 \to 0 \quad \text{for all } \ x, y \in p(\mathbb{C}^n \otimes M) \mathbin{\overline{\otimes}} p(\mathbb{C}^n \otimes M) \ . \quad (2.1)$$

Indeed, (2.1) is obvious when $x = x_1 \otimes x_2$ and $y = y_1 \otimes y_2$ are elementary tensors. Then (2.1) follows easily for general x, y as well. By (2.1) and Theorem 2.3, we have $\Delta(\mathrm{L}\Lambda) \not\prec_{M \overline{\otimes} M} M \mathbin{\overline{\otimes}} P$. Then also the conclusion $\Delta(M_0) \not\prec_{M \overline{\otimes} M} M \mathbin{\overline{\otimes}} P$ follows.

(d) We first state two preliminary observations.

$(*)$ Assume that Q and S are tracial von Neumann algebras and that ${}_{M^r}\mathcal{H}_Q$ and ${}_{\widetilde{M}^r}\mathcal{K}_S$ are bimodules. If the $(M^r \mathbin{\overline{\otimes}} \widetilde{M}^r)$-$(Q \mathbin{\overline{\otimes}} S)$-bimodule $\mathcal{H} \otimes \mathcal{K}$ is left $\Delta(\mathrm{L}\Lambda)$-amenable, then ${}_{\widetilde{M}^r}\mathcal{K}_S$ is left $\mathrm{L}\Lambda$-amenable.

To prove $(*)$, assume that Ω is a $\Delta(\mathrm{L}\Lambda)$-central state on $B(\mathcal{H}\otimes\mathcal{K})\cap(Q^{\mathrm{op}}\,\overline{\otimes}\,S^{\mathrm{op}})'$ whose restriction to $M^r\ \overline{\otimes}\ \widetilde{M}^r$ equals the trace. Then the formula $\Omega_0(T) := \Omega(1\otimes T)$ defines a state on $B(\mathcal{K})\cap(S^{\mathrm{op}})'$ that is $(v_s)_{s\in\Lambda}$-central and whose restriction to $\widetilde{M}^r$ equals the trace. By Lemma 2.11 it follows that Ω_0 is actually $\mathrm{L}\Lambda$-central and this concludes the proof of $(*)$.

$(**)$ Assume that S is a tracial von Neumann algebra and that ${}_{\widetilde{M}}\mathcal{K}_S$ is a bimodule. One can easily check that ${}_{\widetilde{M}}\mathcal{K}_S$ is left M_0-amenable if and only if the bimodule ${}_{\widetilde{M}^r}(p(\mathbb{C}^n\otimes\mathcal{K}))_S$ is left M_0^r-amenable.

We are now ready to prove (d). By our assumptions, the bimodule ${}_{M^r\,\overline{\otimes}\,\widetilde{M}}\mathrm{L}^2(M^r\ \overline{\otimes}\ \widetilde{M})_{M^r\,\overline{\otimes}\,P}$ is left $\Delta(M_0)$-amenable. From $(**)$, we get that

$${}_{M^r\,\overline{\otimes}\,\widetilde{M}^r}\mathrm{L}^2(M^r\ \overline{\otimes}\ p(\mathbb{C}^n\otimes\widetilde{M}))_{M^r\,\overline{\otimes}\,P}$$

is left $\Delta(M_0^r)$-amenable. It then follows from $(*)$ that ${}_{\widetilde{M}^r}(p(\mathbb{C}^n\otimes\mathrm{L}^2(\widetilde{M})))_P$ is left M_0^r-amenable. Again using $(**)$, we get that ${}_{\widetilde{M}}\mathrm{L}^2(\widetilde{M})_P$ is left M_0-amenable, i.e. that M_0 is amenable relative to P inside $\widetilde{M}$.

(e) Assume that ${}_{\mathrm{L}\Lambda}\mathcal{K}_{\mathrm{L}\Lambda}$ is an arbitrary bimodule. Denote by $\lambda:\mathrm{L}(\Lambda)\to B(\mathcal{K})$ and $\rho:(\mathrm{L}\Lambda)^{\mathrm{op}}\to B(\mathcal{K})$ the normal $*$-homomorphisms given by the left, respectively right bimodule action. It is easy to check that there is a unique normal $*$-homomorphism $\Psi:\mathrm{L}\Lambda\ \overline{\otimes}\ (\mathrm{L}\Lambda)^{\mathrm{op}}\to B(\mathcal{K}\otimes\mathcal{K}\otimes\mathcal{K})$ such that

$$\Psi(v_s\otimes v_t^{\mathrm{op}}) = \lambda(v_s)\rho(v_t^{\mathrm{op}})\otimes\lambda(v_s)\otimes\rho(v_t^{\mathrm{op}}),\ \text{for all } s,t\in\Lambda.$$

It follows in particular that the $\mathrm{L}\Lambda$-$\mathrm{L}\Lambda$-bimodule $\mathcal{K}\otimes\mathcal{K}\otimes\mathcal{K}$ given by

$$v_s\cdot(\xi_1\otimes\xi_2\otimes\xi_3)\cdot v_t = (v_s\xi_1 v_t)\otimes(v_s\xi_2)\otimes(\xi_3 v_t)$$

is contained in a multiple of the coarse $\mathrm{L}\Lambda$-$\mathrm{L}\Lambda$-bimodule. Applying this statement to the bimodule ${}_{\mathrm{L}\Lambda}\mathrm{L}^2(M^r)_{\mathrm{L}\Lambda}$, it follows that the $\Delta(\mathrm{L}\Lambda)$-$\Delta(\mathrm{L}\Lambda)$-bimodule

$${}_{\Delta(\mathrm{L}\Lambda)}\big(\mathrm{L}^2(M^r\ \overline{\otimes}\ M^r)\otimes_{M^r\otimes 1}\mathrm{L}^2(M^r\otimes M^r)\big)_{\Delta(\mathrm{L}\Lambda)}$$

is contained in a multiple of the coarse $\Delta(\mathrm{L}\Lambda)$-$\Delta(\mathrm{L}\Lambda)$-bimodule. Then also

$${}_{\Delta(M_0)}\big(\mathrm{L}^2(M^r\ \overline{\otimes}\ M)\otimes_{M^r\otimes 1}\mathrm{L}^2(M^r\otimes M)\big)_{\Delta(M_0)} \tag{2.2}$$

is contained in a multiple of the coarse $\Delta(M_0)$-$\Delta(M_0)$-bimodule.

Assume now that $q\in\Delta(P)'\cap(M\overline{\otimes}M)^r$ is a non-zero projection such that $\Delta(P)q$ is amenable relative to $M^r\otimes 1$. We must prove that P has an amenable direct summand. By our assumption and [PV11, Proposition 2.4.3], the bimodule ${}_{M^r\,\overline{\otimes}\,M}(\mathrm{L}^2(M^r\ \overline{\otimes}\ M)q)_{\Delta(P)q}$ is weakly contained in the bimodule

$${}_{M^r\,\overline{\otimes}\,M}\big(\mathrm{L}^2(M^r\ \overline{\otimes}\ M)\otimes_{M^r\otimes 1}\mathrm{L}^2(M^r\ \overline{\otimes}\ M)q\big)_{\Delta(P)q}\,.$$

Viewing $\mathrm{L}^2(\Delta(P)q)$ as a subspace of $\mathrm{L}^2(M^r \mathbin{\overline{\otimes}} M)q$, it follows that ${}_{\Delta(P)q}\mathrm{L}^2(\Delta(P)q)_{\Delta(P)q}$ is weakly contained in the bimodule

$$ {}_{\Delta(P)q}\big(q\mathrm{L}^2(M^r \mathbin{\overline{\otimes}} M) \otimes_{M^r \otimes 1} \mathrm{L}^2(M^r \otimes M)q\big)_{\Delta(P)q} \; . $$

Since the bimodule in (2.2) is contained in a multiple of the coarse $\Delta(M_0)$-$\Delta(M_0)$-bimodule, we conclude that the trivial $\Delta(P)q$-$\Delta(P)q$-bimodule is weakly contained in the coarse $\Delta(P)q$-$\Delta(P)q$-bimodule. Hence $\Delta(P)$ has an amenable direct summand. Then also P has an amenable direct summand. □

Remark 2.22. Assume that M_0 is a II_1 factor and $r > 0$ such that $M_0^r = \mathrm{L}\Lambda$ for some countable group Λ. Consider the comultiplication

$$ \Delta : \mathrm{L}\Lambda \to \mathrm{L}\Lambda \mathbin{\overline{\otimes}} \mathrm{L}\Lambda : \Delta(v_s) = v_s \otimes v_s \quad \text{for all } \; s \in \Lambda \; . $$

Take a projection $p \in \mathrm{M}_n(\mathbb{C}) \otimes M_0$ with $(\mathrm{Tr} \otimes \tau)(p) = r$ and realize $M_0^r = p(\mathrm{M}_n(\mathbb{C}) \otimes M_0)p$. Realize $(M_0 \mathbin{\overline{\otimes}} M_0)^r$ as $M_0^r \mathbin{\overline{\otimes}} M_0$. The relation between Δ and the amplified comultiplication $\Delta_0 : M_0 \to M_0^r \mathbin{\overline{\otimes}} M_0$ can be concretized in the following slightly painful way.

Denote by $\zeta : \mathrm{M}_n(\mathbb{C}) \otimes M_0 \to M_0 \otimes \mathrm{M}_n(\mathbb{C})$ the flip isomorphism. Put

$$ \Delta_1 := (\mathrm{id} \otimes \mathrm{id} \otimes \zeta^{-1}) \circ (\Delta_0 \otimes \mathrm{id}) \circ \zeta \; , $$

which is a unital $*$-homomorphism from $\mathrm{M}_n(\mathbb{C}) \otimes M_0$ to $M_0^r \mathbin{\overline{\otimes}} \mathrm{M}_n(\mathbb{C}) \mathbin{\overline{\otimes}} M_0$. We then find an element $Z \in M_0^r \mathbin{\overline{\otimes}} \mathrm{M}_n(\mathbb{C}) \mathbin{\overline{\otimes}} M_0$ such that $Z^*Z = \Delta_1(p)$, $ZZ^* = p \otimes p$ and $\Delta(x) = Z\Delta_1(x)Z^*$ for all $x \in M_0^r$.

2.8 Left-right wreath products and inner amenability

We need the following elementary results on left-right wreath products $H^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$, where the direct product group $\Gamma \times \Gamma$ acts on the set Γ by left-right multiplication: $(g,h) \cdot k = gkh^{-1}$. We refer to Section 2.1 for the definition of inner amenability.

Proposition 2.23. *Let H and Γ be arbitrary countable groups with $H \neq \{e\}$. Write $\mathcal{H} := H^{(\Gamma)}$ and consider the left-right wreath product $\mathcal{G} := \mathcal{H} \rtimes (\Gamma \times \Gamma)$. Denote by H_1 the abelianization of H with quotient map $p_1 : H \to H_1$. Define the homomorphism*

$$ p : \mathcal{H} \to H_1 : p(x) = \sum_{g \in \Gamma} p_1(x_g) \; . $$

Denote by $\mathcal{H}_0$ the kernel of p and define $\mathcal{G}_0 := \mathcal{H}_0 \rtimes (\Gamma \times \Gamma)$.

(a) *If Γ is not inner amenable, then $\mathcal{G}$ is also not inner amenable. Even more so, the unitary representation $(\mathrm{Ad}\, g)_{g \in \Gamma \times \Gamma}$ on $\ell^2(\mathcal{G} - \{e\})$ does not have almost invariant vectors. So any subgroup of $\mathcal{G}$ that contains $\Gamma \times \Gamma$ is not inner amenable.*

(b) *If Γ is non-amenable and finitely generated and if Γ has trivial center, then $\mathcal{G}$ is not inner amenable.*

(c) *If Γ is infinite and has trivial center, then $\mathcal{G}_0$ and $\mathcal{G}$ are i.c.c. groups and*

$$(L\mathcal{H}_0)' \cap L\mathcal{G} \subset L\mathcal{H} \quad \textit{and} \quad (L\mathcal{G}_0)' \cap L\mathcal{G} = \mathbb{C}1 \; .$$

Statement (b) in the above proposition is not used in this thesis. We added it in order to put it in contrast with Remark 2.24, where we show that there are non-amenable i.c.c. groups Γ such that $L\mathcal{G}$ is a McDuff II_1 factor, and in particular such that $\mathcal{G}$ is not W*-superrigid, in the sense of Definition 1.1.

Proof. Throughout the proof, we write $G := \Gamma \times \Gamma$. We denote by P_G the orthogonal projection of $\ell^2(\mathcal{G})$ onto $\ell^2(G)$. The action $(\mathrm{Ad}\, g)_{g \in \Gamma \times \{e\}}$ on $\mathcal{G} - G$ has finite stabilizers. Therefore, the restriction of the representation $(\mathrm{Ad}\, g)_{g \in \Gamma \times \{e\}}$ to the invariant subspace $\ell^2(\mathcal{G} - G)$ is (weakly) contained in the regular representation of Γ.

(a) Assume that $\xi_n \in \ell^2(\mathcal{G} - \{e\})$ is a sequence of vectors that is almost invariant under $(\mathrm{Ad}\, g)_{g \in G}$. By the first paragraph and because Γ is non-amenable, it follows that $\|\xi_n - P_G(\xi_n)\|_2 \to 0$. Note that $(P_G(\xi_n))$ is a sequence of vectors in $\ell^2(G - \{e\})$ that is almost invariant under $(\mathrm{Ad}_g)_{g \in G}$. Since Γ is not inner amenable, also G is not inner amenable. Hence $\|P_G(\xi_n)\|_2 \to 0$ and also $\|\xi_n\|_2 \to 0$.

(b) Assume that $\xi_n \in \ell^2(\mathcal{G} - \{e\})$ is a sequence of vectors that is almost invariant under $(\mathrm{Ad}\, g)_{g \in \mathcal{G}}$. By the first paragraph and because Γ is non-amenable, it follows that $\|\xi_n - P_G(\xi_n)\|_2 \to 0$. Fix an element $s \in H - \{e\}$. For every $k \in \Gamma$, denote by $s_k \in H^{(\Gamma)}$ the element s viewed in position k. It is easy to check that $P_G \circ (\mathrm{Ad}\, s_k) \circ P_G = P_{\mathrm{Stab}\, k}$. Since $\|\xi_n - P_G(\xi_n)\| \to 0$ and since the sequence (ξ_n) is almost invariant under $(\mathrm{Ad}\, g)_{g \in \mathcal{G}}$, we conclude that $\|\xi_n - P_{\mathrm{Stab}\, \mathcal{F}}(\xi_n)\| \to 0$ for every finite subset $\mathcal{F} \subset \Gamma$. If $\{k_1, \ldots, k_r\}$ is a finite generating set for Γ, one checks that $\mathrm{Stab}\{e, k_1, \ldots, k_r\} = \{(g,g) \mid g \in Z(\Gamma)\}$. Since Γ has trivial center, we get that $\|\xi_n\| \to 0$.

(c) We start by proving the following claim: for every $g \in G - \{e\}$, there exist infinitely many $k \in \Gamma$ such that $g \cdot k \neq k$. To prove this claim, fix $g = (g_1, g_2) \in G$. The set $S = \{k \in \Gamma \mid g \cdot k = k\}$ is a coset of a subgroup $\Gamma_0 < \Gamma$. So if $\Gamma - S$ is finite, also $\Gamma - \Gamma_0$ is finite and hence empty. Thus $S = \Gamma$

and it follows that $g_1 = g_2$ and that this element belongs to the center of Γ. Since Γ has trivial center, we conclude that $g_1 = g_2 = e$ and hence $g = e$.

Having proven the claim above, we show that for every $x \in \mathcal{G} - \mathcal{H}$, we have that $\{zxz^{-1} \mid z \in \mathcal{H}_0\}$ is infinite. We write $x = yg$ with $y \in \mathcal{H}$ and $g \in G - \{e\}$. Define

$$\mathcal{F}_0 := \{e\} \cup \{g \cdot e\} \cup \{k \in \Gamma \mid y_k \neq e\} \ .$$

By the claim in the previous paragraph, we can inductively choose elements $k_n \in \Gamma$ such that $g \cdot k_n \neq k_n$ for all n and such that the sets $\mathcal{F}_0$, $\{k_1, g \cdot k_1\}$, $\{k_2, g \cdot k_2\}$, ... are all disjoint. Fix an element $s \in H - \{e\}$. For every $k \in \Gamma$, denote by $s_k \in H^{(\Gamma)}$ the element s viewed in position k. Define the sequence of elements $z_n \in \mathcal{H}_0$ given by $z_n := s_e^{-1} s_{k_n}$. Since

$$z_n x z_n^{-1} = s_e^{-1}\, s_{k_n}\, y\, s_{g \cdot k_n}^{-1}\, s_{g \cdot e}\, g \ ,$$

we get that all elements $z_n x z_n^{-1}$ are distinct. So the set $\{zxz^{-1} \mid z \in \mathcal{H}_0\}$ is infinite for every $x \in \mathcal{G} - \mathcal{H}$. This means that $(L\mathcal{H}_0)' \cap L\mathcal{G} \subset L\mathcal{H}$.

It remains to prove that $(L\mathcal{G}_0)' \cap L\mathcal{G} = \mathbb{C}1$. Because of the previous paragraph, it suffices to observe that elements in $\mathcal{H} - \{e\}$ have an infinite conjugacy class under $(\mathrm{Ad}\, g)_{g \in \Gamma \times \{e\}}$. □

Remark 2.24. There are non-amenable i.c.c. groups Γ such that $\mathcal{G} := H^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$ is inner amenable, and even such that $L\mathcal{G}$ is a McDuff II_1 factor (see Section 2.1 for terminology). Indeed, it suffices that Γ admits two sequences of elements $(g_n), (h_n)$ with the property that g_n and h_n do not commute, but eventually commute with any fixed element of Γ. In that case, $u_{(g_n, g_n)}$ and $u_{(h_n, h_n)}$ form two non-commuting central sequences in $L\mathcal{G}$, forcing $L\mathcal{G}$ to be McDuff. Such sequences can be easily found in the i.c.c. group S_∞ of finite permutations of $\mathbb{N}$, and hence also in the non-amenable i.c.c. group $\mathbb{F}_2 \times S_\infty$.

Because of the previous paragraph, *not all non-amenable left-right wreath product groups are W*-superrigid*, in the sense of Definition 1.1.

2.9 Amalgamated free products and HNN extensions

Let Γ_1 and Γ_2 be two countable groups having a common subgroup Σ. We say that the amalgamated free product $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ is *non-degenerate* if we have that $[\Gamma_1 : \Sigma] \geq 2$ and $[\Gamma_2 : \Sigma] \geq 3$. This condition is sufficient to witness the non-amenability of Γ, since Γ contains a copy of $\mathbb{F}_2$.

If Γ_1 is a countable group, $\Sigma < \Gamma_1$ is a subgroup and $\theta : \Sigma \to \Gamma_1$ is an injective group homomorphism, then the HNN extension $\Gamma =$HNN$(\Gamma_1, \Sigma, \theta)$ is the group generated by a copy of Γ_1 and an extra generator t, called stable letter, subject to relations $tgt^{-1} = \theta(g)$, for all $g \in \Sigma$. We say that Γ is *non-degenerate* if $\Sigma \neq \Gamma_1 \neq \theta(\Sigma)$. Also in this case, Γ contains a copy of the free group on two generators, hence it is non-amenable.

The group von Neumann algebra of an HNN extension $\Gamma =$HNN$(\Gamma_1, \Sigma, \theta)$ is precisely the HNN extension of von Neumann algebras HNN$(\mathrm{L}\Gamma_1, \mathrm{L}\Sigma, \Theta)$, associated to the triple $(\mathrm{L}\Gamma_1, \mathrm{L}\Sigma, \Theta)$, where Θ is the trace-preserving embedding $\mathrm{L}\Sigma \to \mathrm{L}\Gamma_1$ induced by θ. For more details about HNN extensions of von Neumann algebras we refer to [Ue05] and [FV10, Section 3].

The next easy lemma is essentially contained in the proof of [Io12b, Theorem 7.1] and [DI12, Lemma 8.2], but we provide a short proof for completeness.

Lemma 2.25. *Let Γ be an amalgamated free product $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ or an HNN extension $\Gamma =$ HNN$(\Gamma_1, \Sigma, \theta)$. If Γ is non-degenerate, then Σ is not co-amenable in Γ.*

Proof. Let $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ and assume that Σ is co-amenable in Γ, i.e. there exists a Γ-invariant state φ on $\ell^\infty(\Gamma/\Sigma)$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the sets of words beginning with a letter in $\Gamma_1 \setminus \Sigma$, respectively $\Gamma_2 \setminus \Sigma$. Then $\Gamma = \mathcal{F}_1 \sqcup \mathcal{F}_2 \sqcup \Sigma$. Let $\pi : \Gamma \to \Gamma/\Sigma$ be the quotient map and define $\mathcal{G}_1 = \pi(\mathcal{F}_1)$ and $\mathcal{G}_2 = \pi(\mathcal{F}_2)$. Thus $\Gamma/\Sigma = \mathcal{G}_1 \sqcup \mathcal{G}_2 \sqcup \{e\Sigma\}$.

Since Γ is non-degenerate, we can take elements $g_1 \in \Gamma_1 \setminus \Sigma$ and $g_2, g_3 \in \Gamma_2 \setminus \Sigma$ such that $g_3^{-1} g_2 \notin \Sigma$. Then we have that $\Sigma \subset g_1\mathcal{F}_1$, $g_1\mathcal{F}_2 \subset \mathcal{F}_1$, $g_2\mathcal{F}_1 \subset \mathcal{F}_2$ and $g_3\mathcal{F}_1 \subset \mathcal{F}_2$, hence $e\Sigma \in g_1\mathcal{G}_1$, $g_1\mathcal{G}_2 \subset \mathcal{G}_1$, $g_2\mathcal{G}_1 \subset \mathcal{G}_2$ and $g_3\mathcal{G}_1 \subset \mathcal{G}_2$.

Since $g_3^{-1} g_2\mathcal{G}_1 \subset \mathcal{G}_2$, then $g_2\mathcal{G}_1 \cap g_3\mathcal{G}_1 = \emptyset$, and hence $g_2\mathcal{G}_1 \sqcup g_3\mathcal{G}_1 \subset \mathcal{G}_2$. For any subset $\mathcal{F} \subset \Gamma/\Sigma$, define $m(\mathcal{F}) := \varphi(\chi_\mathcal{F}) \in [0, 1]$. Then m is a finitely additive Γ-invariant probability measure on Γ/Σ. Since π is Γ-equivariant and m is a finitely additive Γ-invariant measure, it follows that $m(e\Sigma) \leq m(\mathcal{F}_1)$, $m(\mathcal{F}_2) \leq m(\mathcal{F}_1)$ and $2m(\mathcal{F}_1) \leq m(\mathcal{F}_2)$, hence $m(e\Sigma) = m(\mathcal{F}_1) = m(\mathcal{F}_2)$. But this implies that $m(\Gamma/\Sigma) = 0$, which is a contradiction.

Let now $\Gamma =$ HNN$(\Gamma_1, \Sigma, \theta) = \langle \Gamma_1, t \mid tst^{-1} = \theta(s), \forall s \in \Sigma \rangle$ and assume that Σ is co-amenable in Γ, i.e. there exists a Γ-invariant state φ on $\ell^\infty(\Gamma/\Sigma)$. Denote by $\pi : \Gamma \to \Gamma/\Sigma$ the quotient map and, for any subset $\mathcal{F} \subset \Gamma/\Sigma$, define $m(\mathcal{F}) := \varphi(\chi_\mathcal{F})$. Then m is a finitely additive Γ-invariant probability measure on Γ/Σ.

Let $A \subset \Gamma$ and $B \subset \Gamma$ be sets of representatives of left cosets of Σ, respectively $\theta(\Sigma)$ in Γ_1, with $e \in A \cap B$. Since Γ is non-degenerate, we can take elements

$a \in A \setminus \{e\}$ and $b \in B \setminus \{e\}$. By the normal form theorem [LS77, Chapter IV, Theorem 2.1], every element $g \in \Gamma$ has a unique representation $g = g_n t^{\varepsilon_n} g_{n-1} \dots g_1 t^{\varepsilon_1} g_0$, for some $g_0, \dots, g_n \in \Gamma_1$ and $\varepsilon_i \in \{-1, 1\}$ such that $g_i \in A$ if $\varepsilon_i = -1$, $g_i \in B$ if $\varepsilon_i = 1$, for all $i = 1, \dots, n$ and that there are no consecutive subsequences of the form $t^{\varepsilon}, 1, t^{-\varepsilon}$ within the sequence $g_n, t^{\varepsilon_n}, g_{n-1}, \dots, g_1, t^{\varepsilon_1}, g_0$.

Denote by S the set of all such elements $g = g_n t^{\varepsilon_n} g_{n-1} \dots g_1 t^{\varepsilon_1} g_0 \in \Gamma$ such that $n \geq 1$ and $g_n \neq e$ and denote by U, respectively V, the set of all $g \in \Gamma$ such that $n \geq 1$, $g_n = e$ and $\varepsilon_n = -1$, respectively $\varepsilon_n = 1$. Then we have that $t^{-1}S \subset U$, $tS \subset V$, $aU \subset S$, $bV \subset S$ and $aU \cap bV = \emptyset$. Since π is Γ-equivariant and m is a finitely additive Γ-invariant measure, it follows that $m(\pi(S)) = m(\pi(V)) = m(\pi(U)) = 0$. Since $\Gamma/\Sigma = \pi(U) \cup \pi(V) \cup \pi(S) \cup \Gamma_1/\Sigma$, we get that $m(\Gamma_1/\Sigma) = 1$, which is a contradiction since $t(\Gamma_1/\Sigma) \cap \Gamma_1/\Sigma = \emptyset$. $\square$

We end this section with a few results which allow us, under certain malnormality assumptions, to control the centralizers of non-trivial elements in amalgamated free products and HNN extensions.

Let Γ be a countable group. A subgroup $\Sigma < \Gamma$ is called *malnormal* if $\Sigma \cap g\Sigma g^{-1} = \{e\}$, for all $g \in \Gamma \setminus \Sigma$. A subgroup $\Sigma < \Gamma$ is said to be *relatively malnormal* if there exists an infinite index subgroup $\Lambda < \Gamma$ such that $\Sigma \cap g\Sigma g^{-1}$ is finite, for all $g \in \Gamma \setminus \Lambda$. If $\{\Sigma_i\}_{i \in I}$ is a family of subgroups of Γ, then we say that $\{\Sigma_i\}_{i \in I}$ is *malnormal* in Γ if $g\Sigma_i g^{-1} \cap \Sigma_j = \{1\}$, unless $i = j$ and $g \in \Sigma_i$. A subgroup $\Sigma < \Gamma$ is called *weakly malnormal* if there exist $g_1, \dots, g_n \in \Gamma$ such that $\bigcap_{k=1}^n g_k \Sigma g_k^{-1}$ is finite.

Theorem 2.26. *([KS70, Theorem 1] and [Le67, Theorem 2])*

Let $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ be an amalgamated free product and assume that Σ is malnormal in Γ_1. Then the centralizer of any non-trivial element of Γ is either infinite cyclic or contained in a conjugate of Γ_i, for some $i = 1$ or 2.

Moreover, we have that Σ is relatively malnormal in Γ, with respect to the infinite index subgroup Γ_2.

Theorem 2.27. *([KS70, Theorem 9] and its corollaries)*

Let $\Gamma = HNN(\Gamma_1, \Sigma, \theta)$ be an HNN extension and assume that the family $\{\Sigma, \theta(\Sigma)\}$ is malnormal in Γ_1. Then the centralizer of any non-trivial element of Γ is either infinite cyclic or contained in a conjugate of Γ_1.

Moreover, Σ is malnormal in Γ, so in particular, it is relatively malnormal in Γ.

Chapter 3

Spectral gap rigidity for generalized Bernoulli actions

In [Po03], [Po04], Popa discovered his fundamental *malleable deformation* for Bernoulli crossed products $M = A^G \rtimes G$ and used it to establish the first W*-rigidity theorems in the case where the group G has property (T). In [Po06b], Popa introduced his spectral gap methods to prove W*-rigidity theorems for $A^G \rtimes G$ in the case where G is a direct product of non-amenable groups. These methods and results have been generalized in many subsequent works (see e.g. [PV06], [Va07], [Io10], [IPV10]) and were in particular extended to cover certain *generalized* Bernoulli actions, associated with general group actions $G \curvearrowright I$. So far, the spectral gap methods could only be employed under the assumption that the stabilizer of every point $i \in I$ is amenable (see e.g. [IPV10, Corollary 4.3]). In this chapter, we show that it is actually sufficient to have a constant $\kappa > 0$ such that the stabilizer of every finite subset of I, with more than κ points, is amenable.

3.1 The tensor length deformation

Let G be a countable discrete group acting on a countable set I. Assume that (A_0, τ) is an arbitrary tracial von Neumann algebra. We denote by A_0^I the tensor product, with respect to τ, of copies of A_0 indexed by I. We let G act on A_0^I by the generalized Bernoulli action: denoting by $\pi_i : A_0 \to A_0^I$ the embedding of A_0 as the i-th tensor factor, this generalized Bernoulli action

$(\sigma_g)_{g\in G}$ is given by $\sigma_g \circ \pi_i = \pi_{g\cdot i}$ for all $g \in G$ and $i \in I$. We consider the crossed product von Neumann algebra $M := A_0^I \rtimes G$. Whenever $\mathcal{F} \subset I$, we write $\operatorname{Stab}\mathcal{F} := \{g \in G \mid g \cdot i = i, \forall i \in \mathcal{F}\}$.

We use the following variant, due to [Io06], of Popa's malleable deformation for Bernoulli crossed products. Consider the free product $A_0 * L\mathbb{Z}$ with respect to the natural traces. Denote by $\widetilde{M} := (A_0 * L\mathbb{Z})^I \rtimes G$ the corresponding generalized Bernoulli crossed product.

Define the self-adjoint element $h \in L\mathbb{Z}$ with spectrum $[-\pi, \pi]$ such that $\exp(ih)$ equals the canonical generating unitary $u_1 \in L\mathbb{Z}$. Put $u_t := \exp(ith)$ and note that u_t is a one-parameter group of unitaries with $|\tau(u_t)| < 1$ for all $t \neq 0$. As above we denote by $\pi_i : A_0 * L\mathbb{Z} \to (A_0 * L\mathbb{Z})^I$ the embedding as the i-th tensor factor. We can then define the malleable deformation $(\alpha_t)_{t\in\mathbb{R}}$ by automorphisms of $\widetilde{M}$ given by $\alpha_t(u_g) = u_g$ and $\alpha_t(\pi_i(x)) = \pi_i(u_t x u_t^*)$ for all $g \in G$, $t \in \mathbb{R}$, $i \in I$ and $x \in A_0 * L\mathbb{Z}$.

Denote $\rho_t := |\tau(u_t)|^2$ and observe that $0 \leq \rho_t < 1$ for all $t \neq 0$. For every finite subset $\mathcal{F} \subset I$, we denote by $\pi_{\mathcal{F}} : A_0^{\mathcal{F}} \to A_0^I$ the natural embedding. Define the unital completely positive maps $\psi_t : M \to M$ given by $\psi_t(x) = E_M(\alpha_t(x))$ for all $x \in M$. Whenever $a \in A_0^{\mathcal{F}}$ is the elementary tensor given by $a = \underset{i\in\mathcal{F}}{\otimes} a_i$ with $a_i \in A_0 \ominus \mathbb{C}1$, we have

$$\psi_t(\pi_{\mathcal{F}}(a)u_g) = \rho_t^{|F|}\, \pi_{\mathcal{F}}(a)u_g \quad \text{for all } \; t \in \mathbb{R}, g \in G\ .$$

Therefore we consider the malleable deformation $(\alpha_t)_{t\in\mathbb{R}}$, and the corresponding completely positive maps $(\psi_t)_{t\in\mathbb{R}}$, as the *tensor length deformation* of the generalized Bernoulli crossed product $M = A_0^I \rtimes G$.

3.2 Spectral gap rigidity

A unitary representation $(\pi, \mathcal{H})$ of a countable group Γ has *spectral gap* if it does not weakly contain the trivial representation of Γ. If Γ acts on a standard probability space (X, μ) in a measure-preserving way, then we say that the action $\Gamma \curvearrowright (X, \mu)$ has *spectral gap* if the corresponding Koopman representation of Γ on the Hilbert space $\mathrm{L}^2(X) \ominus \mathbb{C}$ has spectral gap. For example, if Γ is non-amenable and acts on a countable set I, with amenable point stabilizers, then the generalized Bernoulli action $\Gamma \curvearrowright (X, \mu)^I$ has spectral gap.

This notion of spectral gap has been considered by Popa in the context of type II_1 factors. More precisely, if M is a type II_1 factor and $Q \subset M$ is a von Neumann subalgebra, then we say that Q has *spectral gap* in M if the adjoint representation of $\mathcal{U}(Q)$ on $\mathrm{L}^2(M) \ominus \mathrm{L}^2(Q' \cap M)$ has spectral gap. Notice that whenever $Q \subset M$ is a subfactor with spectral gap, then automatically Q does not have property Gamma.

Consider the tensor length deformation $(\alpha_t)_{t\in\mathbb{R}} \in \mathrm{Aut}(\widetilde{M})$ constructed above and let $Q \subset \widetilde{M}$ be a von Neumann subalgebra. It is very important for us to detect whether Q is a "rigid" subalgebra or not, in the sense that the deformation α_t converges uniformly to the identity on Q. For instance, if $Q \subset \widetilde{M}$ has property (T), then by definition $\alpha_t \to \mathrm{id}$ uniformly on Q. Another source of getting rigidity is *Popa's spectral gap rigidity principle*. Roughly speaking, Popa's principle says that, under good assumptions, if $Q \subset \widetilde{M}$ has spectral gap, then Q is "rigid". This principle appears in [Po06b, Lemma 5.1], for plain Bernoulli actions, and in [IPV10, Corollary 4.3], for generalized Bernoulli actions.

Theorem 3.1. *Let $G \curvearrowright I$ be an action of a countable group on a countable set. Assume that $\kappa, \ell > 0$ are integers and that $G_1, \ldots, G_\ell < G$ are subgroups with the following property: for every finite subset $\mathcal{F} \subset I$ with $|\mathcal{F}| \geq \kappa$, there exists an $i \in \{1, \ldots, \ell\}$ such that $\mathrm{Stab}\,\mathcal{F}$ is amenable relative to G_i.*

Assume that (A_0, τ) and (N, τ) are arbitrary tracial von Neumann algebras. Consider as above the generalized Bernoulli crossed product $M = A_0^I \rtimes G$ with its tensor length deformation $\alpha_t \in \mathrm{Aut}(\widetilde{M})$.

Assume that $p \in N \overline{\otimes} M$ is a non-zero projection and that $P \subset p(N \overline{\otimes} M)p$ is a von Neumann subalgebra such that for all non-zero projections $q \in P' \cap p(N \overline{\otimes} M)p$ and all $i = 1, \ldots, \ell$, we have that Pq is non-amenable relative to $N \overline{\otimes} (A_0^I \rtimes G_i)$.

Then

$$\sup_{b \in \mathcal{U}(P' \cap p(N \overline{\otimes} M)p)} \|(id \otimes \alpha_t)(b) - b\|_2 \quad \textit{converges to } 0 \textit{ as } t \to 0.$$

Put $\mathcal{M} := N \overline{\otimes} M$ and $\widetilde{\mathcal{M}} := N \overline{\otimes} \widetilde{M}$. The proof of Theorem 3.1 follows closely the proofs of [Po06b, Lemma 5.1] and [IPV10, Corollary 4.3]. The essential difference is that we replace the bimodule ${}_{\mathcal{M}}\mathrm{L}^2(\widetilde{\mathcal{M}} \ominus \mathcal{M})_{\mathcal{M}}$ by the following $\mathcal{M}$-$\mathcal{M}$-submodule

$$\mathcal{K}^\kappa = \overline{\mathrm{span}} \left\{ x \otimes \pi_{\mathcal{F}}(a) u_g \;\middle|\; \begin{array}{l} x \in N,\, g \in G,\, \mathcal{F} \subset I,\, \kappa \leq |\mathcal{F}| < \infty, \\ a = \bigotimes_{i \in \mathcal{F}} a_i \text{ with } a_i \in A_0 * L\mathbb{Z} \text{ for all } i \\ \text{and with } a_i \in A_0 * L\mathbb{Z} \ominus A_0 \text{ for at} \\ \text{least } \kappa \text{ elements } i \in \mathcal{F} \end{array} \right\} . \tag{3.1}$$

Before proving Theorem 3.1, we need the following lemma.

Lemma 3.2. *Under the assumptions of Theorem 3.1, put* $\mathcal{M}_i := N \overline{\otimes} (A_0^I \rtimes G_i)$*. Then there exist* $\mathcal{M}_i$*-*$\mathcal{M}$*-bimodules* $\mathcal{H}_i$ *such that the* $\mathcal{M}$*-*$\mathcal{M}$*-bimodule* $\mathcal{K}^\kappa$ *is weakly contained in the* $\mathcal{M}$*-*$\mathcal{M}$*-bimodule* $\bigoplus_{i=1}^{\ell} \big(L^2(\mathcal{M}) \otimes_{\mathcal{M}_i} \mathcal{H}_i\big)$*.*

Proof. Let $u \in L\mathbb{Z}$ be the canonical generating unitary. Let $\mathcal{A} \subset A_0 \ominus \mathbb{C}1$ be an orthonormal basis of $L^2(A_0) \ominus \mathbb{C}1$. Define $\mathcal{B} \subset A_0 * L\mathbb{Z}$ given by

$$\mathcal{B} := \{u^{n_1} a_1 u^{n_2} a_2 \cdots u^{n_{k-1}} a_{k-1} u^{n_k} \mid k \geq 1,\ n_j \in \mathbb{Z} - \{0\},\ a_j \in \mathcal{A}\} .$$

By construction, we have the following orthogonal decomposition of $L^2(A_0 * L\mathbb{Z})$ into A_0-A_0-subbimodules:

$$L^2(A_0 * L\mathbb{Z}) = L^2(A_0) \oplus \bigoplus_{b \in \mathcal{B}} \overline{A_0 b A_0} .$$

Fix $\mathcal{F} \subset I$ finite, with $|\mathcal{F}| \geq \kappa$, and fix for all $i \in \mathcal{F}$, $c_i \in \mathcal{B}$. Denote

$$c := 1 \otimes \pi_{\mathcal{F}}\Big(\underset{i \in \mathcal{F}}{\otimes} c_i\Big) \in N \,\overline{\otimes}\, (A_0 * L\mathbb{Z})^I .$$

Define the $\mathcal{M}$-$\mathcal{M}$-subbimodule of $\mathcal{K}^\kappa$ given by $\mathcal{K}^c := \overline{\mathcal{M} c \mathcal{M}}$. Define the subgroup $\Lambda < G$ given by

$$\Lambda := \{g \in G \mid g \cdot \mathcal{F} = \mathcal{F}, c_{g \cdot i} = c_i \ \text{ for all } \ i \in \mathcal{F}\} . \tag{3.2}$$

The formula $x \otimes y \mapsto xcy$ defines an $\mathcal{M}$-$\mathcal{M}$-bimodular unitary between $\mathrm{L}^2(\mathcal{M}) \otimes_Q \mathrm{L}^2(\mathcal{M})$ and $\mathcal{K}^c$ with $Q := N \,\overline{\otimes}\, (A_0^{I-\mathcal{F}} \rtimes \Lambda)$. The different $\mathcal{K}^c$ span a dense subspace of $\mathcal{K}^\kappa$. Also, if $\mathcal{F}, c$ and $\mathcal{F}', c'$ are chosen as above, there are two possibilities: either there exists a $g \in G$ such that $\mathcal{F}' = g \cdot \mathcal{F}$ and $c'_{g \cdot i} = c_i$ for all $i \in \mathcal{F}$, or such a $g \in G$ does not exist. In the first case, we have $\mathcal{K}^c = \mathcal{K}^{c'}$, while in the second case, we have $\mathcal{K}^c \perp \mathcal{K}^{c'}$.

Altogether we can choose a sequence of c's as above, denoted c_n, such that $\mathcal{K}^\kappa$ is the orthogonal direct sum of its subbimodules $\mathcal{K}^{c_n}$. To each c_n corresponds a finite subset $\mathcal{F}_n \subset I$ satisfying $|\mathcal{F}_n| \geq \kappa$, and a subgroup $\Lambda_n < G$ given by (3.2). Note that by (3.2), we get that $\operatorname{Stab} \mathcal{F}_n$ is a finite index subgroup of Λ_n. Writing $Q_n = N \,\overline{\otimes}\, (A_0^{I-\mathcal{F}_n} \rtimes \Lambda_n)$, we conclude that $\mathcal{K}^\kappa$ is isomorphic to the direct sum of the sequence of $\mathcal{M}$-$\mathcal{M}$-bimodules $L^2(\mathcal{M}) \otimes_{Q_n} L^2(\mathcal{M})$.

By the assumptions of the lemma, for every n, there exists an $i(n) \in \{1, \ldots, \ell\}$ such that $\operatorname{Stab} \mathcal{F}_n$ is amenable relative to $G_{i(n)}$ inside G. Since $\operatorname{Stab} \mathcal{F}_n < \Lambda_n$ has finite index, also Λ_n is amenable relative to $G_{i(n)}$ inside G. It then follows from Lemma 2.12 that $N \,\overline{\otimes}\, (A_0^I \rtimes \Lambda_n)$ is amenable relative to $\mathcal{M}_{i(n)}$. A fortiori, Q_n is amenable relative to $\mathcal{M}_{i(n)}$. By [PV11, Proposition 2.4.3], this means that ${}_{\mathcal{M}}L^2(\mathcal{M})_{Q_n}$ is weakly contained in ${}_{\mathcal{M}}\big(L^2(\mathcal{M}) \otimes_{\mathcal{M}_{i(n)}} L^2(\mathcal{M})\big)_{Q_n}$. Defining $\mathcal{H}_i$ as the direct sum of all $L^2(\mathcal{M}) \otimes_{Q_n} L^2(\mathcal{M})$ with $i(n) = i$, it follows that $\mathcal{K}^\kappa$ is weakly contained in $\bigoplus_{i=1}^{\ell} \big(L^2(\mathcal{M}) \otimes_{\mathcal{M}_i} \mathcal{H}_i\big)$ as an $\mathcal{M}$-$\mathcal{M}$-bimodule. □

Proof of Theorem 3.1. Denote by $P_{\mathcal{K}^\kappa}$ the orthogonal projection of $\mathrm{L}^2(\widetilde{\mathcal{M}})$ onto the closed subspace $\mathcal{K}^\kappa$ that we defined in (3.1). Denote $\mathcal{U} := \mathcal{U}(P' \cap p(N \mathbin{\overline{\otimes}} M)p)$. We start by proving the following claim that is a variant of Popa's fundamental transversality property in [Po06b, Lemma 2.1].

Claim. If $\sup_{b\in\mathcal{U}} \|P_{\mathcal{K}^\kappa}((\mathrm{id}\otimes\alpha_t)(b))\|_2 \to 0$ when $t \to 0$, then also

$$\sup_{b\in\mathcal{U}} \|(\mathrm{id}\otimes\alpha_t)(b) - b\|_2 \to 0 \text{ when } t \to 0.$$

To prove the claim, we first determine a formula for $\|P_{\mathcal{K}^\kappa}(\mathrm{id}\otimes\alpha_t)(y)\|_2$ when $y \in \mathcal{M}$. For every $n \geq 0$, define the closed subspace $\mathcal{H}_n \subset L^2(\mathcal{M})$ as

$$\mathcal{H}_n := \overline{\mathrm{span}}\left\{ x\otimes\pi_{\mathcal{F}}(a)u_g \;\middle|\; \begin{array}{l} x\in N,\ g\in G,\ \mathcal{F}\subset I \text{ finite, } |\mathcal{F}|=n, \\ a = \bigotimes_{i\in\mathcal{F}} a_i,\ a_i \in A_0\ominus\mathbb{C}1 \text{ for all } i\in\mathcal{F} \end{array} \right\} .$$

Observe that $L^2(\mathcal{M})$ is the orthogonal direct sum of the $\mathcal{H}_n$. Denote by P_n the orthogonal projection of $L^2(\mathcal{M})$ onto $\mathcal{H}_n$.

Fix a finite subset $\mathcal{F} \subset I$ with $|\mathcal{F}| \geq \kappa$ and fix, for all $i \in \mathcal{F}$, elements $a_i \in A_0 \ominus \mathbb{C}1$. Put $a = \bigotimes_{i\in\mathcal{F}} a_i$. For all $x \in N$ and all $g \in G$, we have

$$\begin{aligned} x\otimes\alpha_t(\pi_{\mathcal{F}}(a)u_g) &= x\otimes\pi_{\mathcal{F}}\Big(\bigotimes_{i\in\mathcal{F}} u_t a_i u_t^*\Big)\, u_g \\ &= \sum_{\mathcal{G}\subset\mathcal{F}} x\otimes\ \pi_{\mathcal{G}}\Big(\bigotimes_{i\in\mathcal{G}}(u_t a_i u_t^* - \rho_t a_i)\Big)\, \pi_{\mathcal{F}-\mathcal{G}}\Big(\bigotimes_{i\in\mathcal{F}-\mathcal{G}} \rho_t a_i\Big)\, u_g \ . \end{aligned}$$

In this last sum, the term corresponding to $\mathcal{G} \subset \mathcal{F}$ belongs to $\mathcal{K}^\kappa$ if $|\mathcal{G}| \geq \kappa$, and is orthogonal to $\mathcal{K}^\kappa$ if $|\mathcal{G}| < \kappa$. Therefore, we have for all $x \in N$ and all $g \in G$ that

$$(1-P_{\mathcal{K}^\kappa})(y) = \sum_{\substack{\mathcal{G}\subset\mathcal{F},\\ |\mathcal{G}|<\kappa}} x\otimes\ \pi_{\mathcal{G}}\Big(\bigotimes_{i\in\mathcal{G}}(u_t a_i u_t^* - \rho_t a_i)\Big)\, \pi_{\mathcal{F}-\mathcal{G}}\Big(\bigotimes_{i\in\mathcal{F}-\mathcal{G}} \rho_t a_i\Big)\, u_g \ ,$$

where $y := x\otimes\pi_{\mathcal{F}}(a)u_g$. Assume that $y' = x'\otimes\pi_{\mathcal{F}'}(a')u_{g'}$ is of a similar form.

Since we have that

$$\langle u_t a u_t^* - \rho_t a, u_t b u_t^* - \rho_t b\rangle = (1-\rho_t^2)\,\tau(b^* a) \quad \text{for all } \ a,b\in A_0\ominus\mathbb{C}1 \ ,$$

if follows that

$$\langle (1-P_{\mathcal{K}^\kappa})(\mathrm{id}\otimes\alpha_t)(y), (1-P_{\mathcal{K}^\kappa})(\mathrm{id}\otimes\alpha_t)(y')\rangle = \langle y, y'\rangle \sum_{j=0}^{\kappa-1} \binom{|\mathcal{F}|}{j}\,(1-\rho_t^2)^j\,\rho_t^{2(|\mathcal{F}|-j)} \ ,$$

with both sides being zero if $\mathcal{F} \neq \mathcal{F}'$. We conclude that for all $y \in \mathcal{M}$, we have

$$\|(1 - P_{\mathcal{K}^\kappa})(\mathrm{id} \otimes \alpha_t)(y)\|_2^2 = \sum_{n=0}^{\infty} c_\kappa(t,n)\, \|P_n(y)\|_2^2$$

where

$$c_\kappa(t,n) = \sum_{j=0}^{\min(\kappa-1,n)} \binom{n}{j} (1-\rho_t^2)^j\, \rho_t^{2(n-j)} \; .$$

Note that $c_\kappa(t,n) = 1$ if $n < \kappa$. It follows that

$$\|P_{\mathcal{K}^\kappa}(\mathrm{id} \otimes \alpha_t)(y)\|_2^2 = \sum_{n=0}^{\infty} (1 - c_\kappa(t,n))\, \|P_n(y)\|_2^2 \quad \text{for all } \; y \in \mathcal{M} \; . \tag{3.3}$$

To prove the claim, assume that

$$\sup_{b \in \mathcal{U}} \|P_{\mathcal{K}^\kappa}(\mathrm{id} \otimes \alpha_t)(b)\|_2 \to 0 \quad \text{when} \quad t \to 0 \; .$$

Choose $\varepsilon > 0$. Take $t > 0$ such that $\|P_{\mathcal{K}^\kappa}(\mathrm{id} \otimes \alpha_t)(b)\|_2 < \varepsilon$ for all $b \in \mathcal{U}$. Since $c_\kappa(t,n) \to 0$ when $n \to \infty$ and t is fixed, we can take n_0 such that $c_\kappa(t,n) < 1/2$ for all $n \geq n_0$. It then follows from (3.3) that for all $b \in \mathcal{U}$, we have

$$\varepsilon^2 > \|P_{\mathcal{K}^\kappa}(\mathrm{id} \otimes \alpha_t)(b)\|_2^2 \geq \frac{1}{2} \sum_{n=n_0}^{\infty} \|P_n(b)\|_2^2 \; . \tag{3.4}$$

We finally take $s_0 > 0$ such that $1 - \rho_s^n < \varepsilon^2$ for all $|s| < s_0$ and all $0 \leq n < n_0$. Using (3.4), it follows that for all $b \in \mathcal{U}$ and all $|s| < s_0$, we have

$$\begin{aligned}
\|(\mathrm{id} \otimes \alpha_s)(b) - b\|_2^2 &= \sum_{n=0}^{\infty} 2(1 - \rho_s^n)\, \|P_n(b)\|_2^2 \\
&\leq \sum_{n=0}^{n_0-1} 2\varepsilon^2\, \|P_n(b)\|_2^2 + 2 \sum_{n=n_0}^{\infty} \|P_n(b)\|_2^2 \\
&\leq 2\varepsilon^2 + 4\varepsilon^2 \; .
\end{aligned}$$

So, $\|(\mathrm{id} \otimes \alpha_s)(b) - b\|_2 \leq 3\varepsilon$ for all $|s| < s_0$ and all $b \in \mathcal{U}$. This proves the claim.

To prove the theorem, assume that $\sup\{\|(\mathrm{id} \otimes \alpha_t)(b) - b\|_2 \mid b \in \mathcal{U}\}$ does not tend to 0 as $t \to 0$. We will produce a non-zero projection $q \in P' \cap p\mathcal{M}p$ and a $j \in \{1, \ldots, \ell\}$ such that Pq is amenable relative to $\mathcal{M}_j$. This will conclude the proof of the theorem.

By the claim above, we find an $\varepsilon > 0$, a $t_0 > 0$, and for every $0 < t < t_0$, a unitary $b_t \in \mathcal{U}$ such that $\|P_{\mathcal{K}^\kappa}(\mathrm{id} \otimes \alpha_t)(b_t)\|_2 \geq \varepsilon$. Define $\xi_t := P_{\mathcal{K}^\kappa}(\mathrm{id} \otimes \alpha_t)(b_t)$. We have $\|\xi_t\|_2 \geq \varepsilon$ for all $0 < t < t_0$. For every fixed $x \in P$, we claim that $\|x\xi_t - \xi_t x\|_2 \to 0$ as $t \to 0$. Indeed, since $P_{\mathcal{K}^\kappa}$ is $\mathcal{M}$-$\mathcal{M}$-bimodular, we get that

$$\begin{aligned} \|x\xi_t - \xi_t x\|_2 &\leq \|x\,(\mathrm{id} \otimes \alpha_t)(b_t) - (\mathrm{id} \otimes \alpha_t)(b_t)\,x\|_2 \\ &= \|(\mathrm{id} \otimes \alpha_{-t})(x)\,b_t - b_t\,(\mathrm{id} \otimes \alpha_{-t})(x)\|_2\ . \end{aligned}$$

Since $b_t \in \mathcal{U}$ is a unitary that commutes with $x \in P$, we conclude that

$$\begin{aligned} \|x\xi_t - \xi_t x\|_2 &\leq 2\|(\mathrm{id} \otimes \alpha_{-t})(x) - x\|_2 + \|x\,b_t - b_t\,x\|_2 \\ &= 2\|(\mathrm{id} \otimes \alpha_{-t})(x) - x\|_2\ . \end{aligned}$$

For a fixed $x \in P$, the last expression tends to zero as $t \to 0$. This proves our claim that $\|x\xi_t - \xi_t x\|_2 \to 0$ as $t \to 0$.

Since $P_{\mathcal{K}^\kappa}$ is $\mathcal{M}$-$\mathcal{M}$-bimodular and b_t is unitary, we also have that $\|x\xi_t\|_2 \leq \|x\|_2$ for all $x \in \mathcal{M}$. So Lemma 2.18 provides a non-zero projection $q \in P' \cap p\mathcal{M}p$ such that the $q\mathcal{M}q$-$\mathcal{M}$-bimodule $q\mathcal{K}^\kappa$ is left Pq-amenable. Using [PV11, Corollary 2.5] and Lemma 3.2, we find $\mathcal{M}_j$-$\mathcal{M}$-bimodules $\mathcal{H}_j$ such that $\bigoplus_{j=1}^{\ell} qL^2(\mathcal{M}) \otimes_{\mathcal{M}_j} \mathcal{H}_j$ is left Pq-amenable. Making $q \in P' \cap p\mathcal{M}p$ smaller, Lemma 2.19 yields a $j \in \{1, \ldots, \ell\}$ such that $qL^2(\mathcal{M}) \otimes_{\mathcal{M}_j} \mathcal{H}_j$ is a left Pq-amenable bimodule. By [PV11, Proposition 2.4.4], the $q\mathcal{M}q$-$\mathcal{M}_j$-bimodule $qL^2(\mathcal{M})$ is left Pq-amenable. This precisely means that Pq is amenable relative to $\mathcal{M}_j$. □

We also need the following variant of [Po03, Theorem 4.1] and its subsequent generalizations in [Io10, Theorem 2.1] and [IPV10, Theorem 4.2]. Since our proof is almost identical, we are rather brief.

Theorem 3.3. *Let $G \curvearrowright I$ be an action of a countable group on a countable set. Assume that (A_0, τ) and (N, τ) are arbitrary tracial von Neumann algebra. Consider as above the generalized Bernoulli crossed product $M = A_0^I \rtimes G$ with its tensor length deformation $\alpha_t \in \mathrm{Aut}(\widetilde{M})$.*

Assume that $p \in N \mathbin{\overline{\otimes}} M$ is a non-zero projection and that $Q \subset p(N \mathbin{\overline{\otimes}} M)p$ is a von Neumann subalgebra generated by a group of unitaries $\mathcal{G} \subset \mathcal{U}(Q)$ with the property that

$$\sup_{b \in \mathcal{G}} \|(id \otimes \alpha_t)(b) - b\|_2 \quad \textit{converges to } 0 \textit{ as } t \to 0.$$

If G is icc, if N is a factor and if for all $i \in I$, we have that $Q \not\prec N \overline{\otimes} (A_0^I \rtimes \operatorname{Stab} i)$, then there exists a partial isometry $v \in N \overline{\otimes} M$ with $vv^* = p$ and $v^* Q v \subset N \overline{\otimes} LG$.

Proof. As above, we put $\mathcal{M} = N \overline{\otimes} M$ and $\widetilde{\mathcal{M}} = N \overline{\otimes} \widetilde{M}$. We first prove the existence of a non-zero partial isometry $v \in \mathcal{M}$ with the properties that $vv^* \in Q' \cap p\mathcal{M}p$ and that $v^* Q v \subset N \overline{\otimes} LG$. We reason exactly as in the proofs of [Po03, Theorem 4.1], [Io10, Theorem 2.1] and [IPV10, Theorem 4.2]. For completeness, we nevertheless provide some details.

By the uniform convergence of $\mathrm{id} \otimes \alpha_t$ on $\mathcal{G}$, we find a $t > 0$ and a non-zero partial isometry $w_0 \in p\widetilde{\mathcal{M}}(\mathrm{id} \otimes \alpha_t)(p)$ such that $x w_0 = w_0 (\mathrm{id} \otimes \alpha_t)(x)$ for all $x \in Q$. We may assume that t is of the form $t = 2^{-n}$. Since for all $i \in I$, we have that $Q \not\prec N \overline{\otimes} (A_0^I \rtimes \operatorname{Stab} i)$, it follows from [IPV10, Lemma 4.1.1] that $w_0 w_0^* \in \mathcal{M}$ and $w_0^* w_0 \in (\mathrm{id} \otimes \alpha_t)(\mathcal{M})$. Define the period two automorphism $\beta \in \operatorname{Aut}(\widetilde{M})$ given by $\beta(x) = x$ for all $x \in M$ and $\beta(\pi_i(u_1)) = u_1^*$ for all $i \in I$. By construction, we have that $\beta \circ \alpha_t = \alpha_{-t} \circ \beta$.

We can now define

$$w_1 := (\mathrm{id} \otimes \alpha_t)((\mathrm{id} \otimes \beta)(v^*) v)$$

and check that w_1 is a non-zero partial isometry in $p\widetilde{\mathcal{M}}(\mathrm{id} \otimes \alpha_{2t})(p)$ satisfying $x w_1 = w_1 (\mathrm{id} \otimes \alpha_{2t})(x)$ for all $x \in Q$. Continuing inductively, we find a non-zero partial isometry $w \in p\widetilde{\mathcal{M}}(\mathrm{id} \otimes \alpha_1)(p)$ satisfying $xw = w(\mathrm{id} \otimes \alpha_1)(x)$ for all $x \in Q$. Literally repeating a part of the proof of [IPV10, Theorem 4.2], we find a finite, possibly empty, subset $\mathcal{F} \subset I$ such that $Q \prec N \overline{\otimes} (A_0^{\mathcal{F}} \rtimes \operatorname{Stab} \mathcal{F})$. Our assumption that $Q \not\prec N \overline{\otimes} (A_0^I \rtimes \operatorname{Stab} i)$ for all $i \in I$, ensures that $\mathcal{F} = \emptyset$. So, $Q \prec N \overline{\otimes} LG$.

Take $n \in \mathbb{N}$, a non-zero partial isometry $v \in \mathrm{M}_{1,n}(\mathbb{C}) \overline{\otimes} p(N \overline{\otimes} M)p$, a projection $q \in \mathrm{M}_n(\mathbb{C}) \overline{\otimes} N \overline{\otimes} LG$ and a $*$-homomorphism $\theta : Q \to q(\mathrm{M}_n(\mathbb{C}) \overline{\otimes} N \overline{\otimes} LG)q$ such that $xv = v\theta(x)$ for all $x \in Q$. Since $Q \not\prec N \overline{\otimes} (A_0^I \rtimes \operatorname{Stab} i)$ for all $i \in I$, by [Va07, Remark 3.8], we may assume that for all $i \in I$, we have $\theta(Q) \not\prec N \overline{\otimes} L(\operatorname{Stab} i)$. By [IPV10, Lemma 4.1.1], we then get that

$$\theta(Q)' \cap q(\mathrm{M}_n(\mathbb{C}) \overline{\otimes} N \overline{\otimes} M)q \subset \mathrm{M}_n(\mathbb{C}) \overline{\otimes} N \overline{\otimes} LG \; .$$

In particular, $v^* v$ is a projection in $\mathrm{M}_n(\mathbb{C}) \overline{\otimes} N \overline{\otimes} LG$ of trace at most 1. Since $N \overline{\otimes} LG$ is a II_1 factor, we may then assume that $n = 1$. So, we have found a non-zero partial isometry $v \in \mathcal{M}$ with the properties that $vv^* \in Q' \cap p\mathcal{M}p$ and that $v^* Q v \subset N \overline{\otimes} LG$.

Let v_n be a maximal sequence of non-zero partial isometries $v_n \in \mathcal{M}$ with the property that the $v_n v_n^*$ are orthogonal projections in $Q' \cap p\mathcal{M}p$ such that $v_n^* Q v_n \subset N \overline{\otimes} LG$. Put $p_0 := p - \sum_n v_n v_n^*$. Since we can apply the previous paragraph to $Qp_0 \subset p_0\mathcal{M}p_0$, the maximality of the sequence (v_n) ensures us that $p_0 = 0$.

Since $N \mathbin{\overline{\otimes}} LG$ is a II_1 factor and since the $v_n^* v_n$ form a sequence of projections in $N \mathbin{\overline{\otimes}} LG$ with $\sum_n v_n v_n^* = p$, we can take partial isometries $w_n \in N \mathbin{\overline{\otimes}} LG$ such that $w_n w_n^* = v_n^* v_n$ for all n and such that the projections $w_n^* w_n$ are orthogonal. Then $v := \sum_n v_n w_n$ is a partial isometry in $\mathcal{M}$ with $vv^* = p$ and $v^* Q v \subset N \mathbin{\overline{\otimes}} LG$. $\square$

Chapter 4

Cocycles and Gaussian deformation

We have pointed out in the introduction that (cf. [BV97], [PT07]) a countable group Γ has positive first ℓ^2-Betti number if and only if it is non-amenable and admits an unbounded 1-cocycle into the left regular representation. Therefore, it is important for us to study 1-cocycles associated to orthogonal representations of Γ. To make this more precise, let us assume that $\Gamma \curvearrowright (B, \tau)$ is a trace-preserving action of Γ on a tracial von Neumann algebra (B, τ) and denote by M the associated crossed product. In this chapter, given an orthogonal representation π of Γ and a 1-cocycle c on Γ associated to π, we construct a *malleable deformation* of M (in the sense of Popa) by automorphisms, i.e. a canonical larger von Neumann algebra $\widetilde{M} \supset M$ together with a one-parameter group of automorphisms $(\beta_t)_{t \in \mathbb{R}}$ of $\widetilde{M}$ such that β_t converges to the identity pointwise, as $t \to 0$, in the L^2-norm on $\widetilde{M}$.

4.1 Cocycles on countable groups

Let Γ be a countable group and let $\pi : \Gamma \to \mathcal{O}(\mathcal{H}_{\mathbb{R}})$ be an orthogonal representation of Γ on a real separable Hilbert space $\mathcal{H}_{\mathbb{R}}$. A 1-*cocycle* on Γ associated to π is a map $c : \Gamma \to \mathcal{H}_{\mathbb{R}}$ satisfying the following 1-cocycle relation

$$c(gh) = c(g) + \pi(g)c(h), \text{ for all } g, h \in \Gamma.$$

The function $\Gamma \ni g \mapsto \|c(g)\|^2 \in \mathbb{C}$ is conditionally of negative type, so, by the Schoenberg's theorem, the 1-cocycle c defines a one-parameter family $(\psi_t)_{t>0}$ of positive definite functions on Γ by

$$\psi_t : \Gamma \ni g \longmapsto \psi_t(g) := \exp(-t\,\|c(g)\|^2) \in \mathbb{R}.$$

If $\Gamma \curvearrowright (B,\tau)$ is a trace-preserving action of Γ on the tracial von Neumann algebra (B,τ) and $M = B \rtimes \Gamma$ is the corresponding crossed product, then to the family $(\psi_t)_{t>0}$ corresponds a one-parameter family $(\varphi_t)_{t>0}$ of unital completely positive normal trace-preserving maps on M, defined by

$$\varphi_t : M \ni bu_g \longmapsto \varphi_t(bu_g) := \psi(g)bu_g = \exp(-t\,\|c(g)\|^2)bu_g \in M.$$

If $\mathcal{H}_\mathbb{R}$ is a real Hilbert space and $\pi : \Gamma \to \mathcal{O}(\mathcal{H}_\mathbb{R})$ is an orthogonal representation, then we denote by $\mathcal{H}$ the complexification of $\mathcal{H}_\mathbb{R}$ and by π the corresponding unitary representation on $\mathcal{H}$. Recall from Section 2.2 that to any unitary representation $\pi : \Gamma \to B(\mathcal{H})$ we associate the M-M-bimodule $\mathcal{H}^\pi := \mathrm{L}^2(M)\otimes\mathcal{H}$, where the left-right M-module action on $\mathcal{K}^\pi$ is given by

$$(bu_g)\cdot(x\otimes\xi)\cdot y = (bu_g)xy \otimes \pi(g)\xi,$$

for all $b \in B, g \in \Gamma, \xi \in \mathcal{H}$ and $x, y \in M$.

Lemma 4.1 ([Io11, Lemma 2.5])**.** *Let $\pi : \Gamma \to \mathcal{O}(\mathcal{H}_\mathbb{R})$ be an orthogonal representation of Γ and $c : \Gamma \to \mathcal{H}_\mathbb{R}$ be a 1-cocycle associated to π. Let $\Lambda < \Gamma$ be a non-amenable subgroup.*

1. *If π is weakly contained in the left regular representation λ of Γ, then the restriction of c to the centralizer of Λ is bounded.*
2. *Suppose that π is mixing and that $c(g) = \lambda(g)\xi - \xi$, for some $\xi \in \ell^2(\Gamma)$ and for all $g \in \Lambda$. Let $h \in \Gamma$ be such that $h\Lambda h^{-1} \cap \Lambda$ is infinite. Then we have that $c(h) = \lambda(h)\xi - \xi$.*

Proof. 1. Since Λ is non-amenable, the restriction of π to Λ does not have almost invariant vectors. Thus we can find elements $s_1, \dots, s_n \in \Lambda$ such that

$$\|\xi\| \le \sum_{i=1}^{n} \|\pi(s_i)\xi - \xi\|, \quad \text{for all } \xi \in \ell^2(\Gamma).$$

If $g \in \Gamma$ belongs to the centralizer of Λ, then

$$\|c(g)\| \le \sum_{i=1}^{n} \|\pi(s_i)c(g) - c(g)\| = \sum_{i=1}^{n} \|\pi(g)c(s_i) - c(s_i)\| \le 2\sum_{i=1}^{n} \|c(s_i)\|.$$

2. Define a new cocycle c_0 on Γ by letting $c_0(g) = c(g) - (\pi(g)\xi - \xi)$, for all $g \in \Gamma$. Then $c_0(s) = 0$, for all $s \in \Lambda$. If $h \in \Gamma$ is such that $h\Lambda h^{-1} \cap \Lambda$ is infinite, then fix $g \in h\Lambda h^{-1} \cap \Lambda$ and let $k \in \Lambda$ be such that $gh = hk$. Since $c_0(g) = c_0(k) = 0$, it follows that $\pi(g)c_0(h) = c_0(h)$, for all $g \in h\Lambda h^{-1} \cap \Lambda$. But since π is mixing, we get that $c_0(h) = 0$.

$\square$

4.2 Gaussian spaces and actions

Let Γ be a countable group and let $\pi : \Gamma \to \mathcal{O}(\mathcal{H}_{\mathbb{R}})$ be an orthogonal representation of Γ on a separable real Hilbert space $\mathcal{H}_{\mathbb{R}}$. We briefly describe the construction of the standard Gaussian probability space and the Gaussian action associated to π. For detailed constructions we refer to [Ke10] and [PS09].

If $\mathcal{H}_{\mathbb{R}}$ is a separable real Hilbert space, then one can prove that there exists a standard probability space (X, μ) together with a set of random variables $(L_\xi)_{\xi \in \mathcal{H}_{\mathbb{R}}}$ such that:

- For every $\xi_1, \ldots, \xi_n \in \mathcal{H}_{\mathbb{R}}$, the random variable $x \mapsto (L_{\xi_1}(x), \ldots, L_{\xi_n}(x))$ has the multivariate normal distribution with mean 0 and covariance matrix $(\langle \xi_i, \xi_j \rangle)_{1 \leq i,j \leq n}$.
- The random variables $(L_\xi)_{\xi \in \mathcal{H}_{\mathbb{R}}}$ generate the Borel σ-algebra.

If another standard probability space (Y, ν), together with random variables $(R_\xi)_{\xi \in \mathcal{H}_{\mathbb{R}}}$, satisfy the same properties, there is an a.e. unique p.m.p. isomorphism
$\Delta : (X, \mu) \to (Y, \nu)$ such that for all $\xi \in \mathcal{H}_{\mathbb{R}}$ we have that $R_\xi = L_\xi \circ \Delta^{-1}$ a.e.

In particular, for every orthogonal transformation $u \in \mathcal{O}(\mathcal{H}_{\mathbb{R}})$ there is an a.e. unique p.m.p. automorphism Δ_u of (X, μ) such that for all $\xi \in \mathcal{H}_{\mathbb{R}}$ we have $L_{u\xi} = L_\xi \circ \Delta_u^{-1}$ a.e.

If $\pi : \Gamma \to \mathcal{O}(\mathcal{H}_{\mathbb{R}})$ is an orthogonal representation, then every element $g \in \Gamma$ defines an orthogonal transformation $\pi(g)$ of $\mathcal{H}_{\mathbb{R}}$ and hence there exists an a.e. unique p.m.p. automorphism Δ_g of (X, μ) such that for all $\xi \in \mathcal{H}_{\mathbb{R}}$ we have $L_{\pi(g)\xi} = L_\xi \circ \Delta_g^{-1}$ a.e. Thus π gives rise to an action $(\Delta_g)_{g \in \Gamma}$ of Γ on the Gaussian probability space (X, μ) by p.m.p. automorphisms. For every $g \in \Gamma$, denote by $\sigma_g \in \mathcal{O}(\mathrm{L}^2_{\mathbb{R}}(X, \mu))$ the corresponding orthogonal transformation defined by $\sigma_g(F) = F \circ \Delta_g^{-1}$. By construction, $\sigma_g(L_\xi) = L_{\pi(g)\xi}$, for all $g \in \Gamma$

and $\xi \in \mathcal{H}_{\mathbb{R}}$. Moreover, one can prove that the corresponding Koopman representation of Γ on $\mathrm{L}^2_{\mathbb{R}}(X,\mu) \ominus \mathbb{C}1$ is equivalent to the direct sum of the symmetric tensor powers $\pi_{\mathrm{sym}}^{\otimes n}$, $n \geq 1$.

To make the last sentence more precise, let us denote by $\mathcal{H}^{\otimes n}$ the n-fold tensor power of $\mathcal{H}_{\mathbb{R}}$. The symmetric group S_n acts obviously on $\mathcal{H}^{\otimes n}$ and we denote by $\mathcal{H}_{\mathrm{sym}}^{\otimes n}$ the closed subspace of all S_n-invariant vectors in $\mathcal{H}^{\otimes n}$. Denote by P_{sym} the orthogonal projection $\mathcal{H}^{\otimes n} \to \mathcal{H}_{\mathrm{sym}}^{\otimes n}$. Define an orthogonal representation $\pi_{\mathrm{sym}}^{\otimes n}$ of Γ on $\mathcal{H}_{\mathrm{sym}}^{\otimes n}$ by letting $\pi_{\mathrm{sym}}^{\otimes n}(g) = P_{\mathrm{sym}} \circ \pi^{\otimes n}(g)$, for all $g \in \Gamma$. Then $\pi_{\mathrm{sym}} := \oplus_{n=0}^{\infty} \pi_{\mathrm{sym}}^{\otimes n}$ is an orthogonal representation of Γ on the symmetric full Fock space $\mathcal{F}_{\mathrm{sym}}(\mathcal{H}_{\mathbb{R}}) := \bigoplus_{n=0}^{\infty} \mathcal{H}_{\mathrm{sym}}^{\otimes n}$ and one can prove that there exists a unitary $V : \mathcal{F}_{\mathrm{sym}}(\mathcal{H}_{\mathbb{R}}) \to \mathrm{L}^2_{\mathbb{R}}(X,\mu)$ intertwining these two representations.

For every $\xi \in \mathcal{H}_{\mathbb{R}}$, denote by $a(\xi)$ the annihilation operator on $\mathcal{F}_{\mathrm{sym}}(\mathcal{H}_{\mathbb{R}})$ and by $a^*(\xi)$ the creation operator on $\mathcal{F}_{\mathrm{sym}}(\mathcal{H}_{\mathbb{R}})$. They satisfy the canonical commutation relations (CCR), i.e. $[a(\xi), a(\eta)] = 0$, $[a^*(\xi), a^*(\eta)] = 0$ and $[a(\xi), a^*(\eta)] = \langle \xi, \eta \rangle 1$, for all $\xi, \eta \in \mathcal{H}_{\mathbb{R}}$.

For any $\xi \in \mathcal{H}_{\mathbb{R}}$, define the self-adjoint operator $b(\xi) := a(\xi) + a^*(\xi)$ and denote by $D \subset B(\mathcal{F}_{\mathrm{sym}}(\mathcal{H}_{\mathbb{R}}))$ the von Neumann algebra generated by the family of unitaries $\{\omega(\xi) := \exp(2\pi i b(\xi)) \mid \xi \in \mathcal{H}_{\mathbb{R}}\}$. Notice that, by the CCR, D is an abelian von Neumann algebra and for all $g \in \Gamma$ and $\xi \in \mathcal{H}_{\mathbb{R}}$ we have that

$$\pi_{\mathrm{sym}}(g) b(\xi) \pi_{\mathrm{sym}}(g)^* = b(\pi(g)\xi). \tag{4.1}$$

One can check that the vacuum vector $\Omega \in \mathcal{F}_{\mathrm{sym}}(\mathcal{H}_{\mathbb{R}})$ (i.e. $\mathcal{H}_{\mathrm{sym}}^{\otimes 0} = \mathbb{C}\Omega$) is cyclic and tracial for D and hence, we may identify $\mathcal{F}_{\mathrm{sym}}(\mathcal{H}_{\mathbb{R}})$ with $\mathrm{L}^2(D,\tau)$, where τ is the trace defined by Ω. Since π_{sym} and σ are unitarily equivalent and using the formula (4.1), we may moreover identify (D,τ) with $\mathrm{L}^\infty(X,\mu)$ and define a trace-preserving action $\sigma : \Gamma \curvearrowright (D,\tau)$ by letting

$$\sigma_g(\omega(\xi)) = \omega(\pi(g)\xi), \text{ for all } g \in \Gamma \text{ and } \xi \in \mathcal{H}_{\mathbb{R}}.$$

This action is called the *Gaussian action* associated to the orthogonal representation π.

4.3 The Gaussian deformation

Let Γ be a countable group, $\pi : \Gamma \to \mathcal{O}(\mathcal{H}_{\mathbb{R}})$ be an orthogonal representation and $c : \Gamma \to \mathcal{H}_{\mathbb{R}}$ be a 1-cocycle for π. Consider the Gaussian action $\sigma : \Gamma \curvearrowright (D,\tau)$

associated to π. Let $\alpha : \Gamma \curvearrowright (B, \tau)$ be a trace-preserving action of Γ on a tracial von Neumann algebra and denote by $M = B \rtimes \Gamma$ the crossed product.

We are now ready to define the *malleable Gaussian deformation* (cf. [Si10, Section 3]) of M, corresponding to the 1-cocycle $c : \Gamma \to \mathcal{H}_{\mathbb{R}}$ into the orthogonal representation $\pi : \Gamma \to \mathcal{O}(\mathcal{H}_{\mathbb{R}})$.

We denote $\widetilde{M} := (D \mathbin{\overline{\otimes}} B) \rtimes \Gamma$, where Γ acts diagonally on $D \mathbin{\overline{\otimes}} B$, and we define a one-parameter group of automorphisms $(\beta_t)_{t\in\mathbb{R}} \in \mathrm{Aut}(\widetilde{M})$ by

$$\beta_t(x) = x, \text{ for all } x \in D \mathbin{\overline{\otimes}} B,$$

and

$$\beta_t(u_g) = (\omega(tc(g)) \otimes 1)u_g, \quad \text{for all } g \in \Gamma, t \in \mathbb{R}.$$

The automorphisms $(\beta_t)_{t\in\mathbb{R}} \in \mathrm{Aut}(\widetilde{M})$ give a malleable deformation in the sense of Popa, i.e. $\beta_t \to \mathrm{id}$ pointwise, as $t \to 0$, in the L^2-norm on $\widetilde{\mathcal{M}}$.

If $Q \subset M$ is a von Neumann subalgebra, then there exists a unique maximal projection $q \in Q' \cap M$ such that the Gaussian deformation β_t converges uniformly to the identity on the unit ball of Qq. Moreover, one can prove that to establish the uniform convergence of β_t on the unit ball of Qq, it suffices to prove the uniform convergence on $r\mathcal{G}r$, where $r \leq q$ is a smaller projection and $\mathcal{G}$ is a group of unitaries generating the subalgebra Q. The precise formulation of these results goes as follows.

Lemma 4.2 ([Va10b, Lemma 3.3])**.** *Let $p \in M$ be a projection and $Q \subset pMp$ be a von Neumann subalgebra. Denote by P the normalizer of Q inside pMp. Then the set of projections*

$$\mathcal{P} = \{q_0 \in Q' \cap pMp \mid \beta_t \to id \text{ uniformly on } (Qq_0)_1\}$$

attains its maximum in a unique projection $q \in \mathcal{P}$, which belongs to the center of P.

Lemma 4.3 ([Va10b, Lemma 3.4])**.** *Let $p \in M$ be a projection and $Q \subset pMp$ be a von Neumann subalgebra generated by a group of unitaries $\mathcal{G} \subset \mathcal{U}(B)$. Let $r \in pMp$ be any projection such that $\beta_t \to id$ uniformly on the set $r\mathcal{G}r$.*

Denote by P the normalizer of Q inside pMp and by q the smallest projection in the center of P satisfying $r \leq q$. Then $\beta_t \to id$ uniformly on the unit ball of Qq.

In [Pe09, Theorem 4.5] and [CP10, Theorem 2.5], using Peterson's techniques of unbounded derivations, it has been proven that whenever π is mixing and

$\beta_t \to \text{id}$ uniformly on a von Neumann subalgebra $Q \subset M$ such that $Q \not\prec B$, then $\beta_t \to \text{id}$ uniformly on the normalizer of Q. An alternative proof of this result was given by Vaes, in [Va10b], using the Gaussian automorphisms $(\beta_t)_{t\in\mathbb{R}}$. The precise formulation of this result goes as follows.

Theorem 4.4 ([Va10b, Theorem 3.10])**.** *Assume that π is a mixing representation. Let $p \in M$ be a projection and $Q \subset pMp$ be a von Neumann subalgebra such that $Q \not\prec B$ and such that $\beta_t \to id$ uniformly on the unit ball of Qq, for some non-zero projection $q \in Q' \cap pMp$. Denote by P the normalizer of Q inside pMp. Then $\beta_t \to id$ uniformly on the unit ball of Pr, where r is the smallest central projection in $Z(P)$ satisfying $q \leq r$.*

Using the same techniques, Chifan and Peterson proved in [CP10, Theorem 3.2] also a converse result, namely if $Q \subset M$ is an abelian von Neumann subalgebra that is normalized by a "large" sequence of unitaries in M on which the deformation β_t converges uniformly to the identity, then β_t converges uniformly to the identity on the unit ball of Q. We will not use this result in the thesis, but we nevertheless state it to make the picture complete. The following formulation is as in [Va10b, Theorem 4.1].

Theorem 4.5. *Assume that π is a mixing representation. Let $p \in M$ be a projection and $Q \subset pMp$ be an abelian von Neumann subalgebra that is normalized by a sequence of unitaries $(u_n)_n \subset \mathcal{U}(pMp)$. Assume that $\|E_Q(xu_ny)\|_2 \to 0$, for all $x, y \in pMp$ and that $\beta_t \to id$ uniformly on $(u_n)_n$.*

Then $\beta_t \to id$ uniformly on the unit ball of Q.

We end this section with two easy lemmas that we will use later on.

Lemma 4.6 ([Io11, Lemma 2.1])**.** *If $\beta_t \to id$ uniformly on the unit ball of pMp, for some non-zero projection $p \in M$, then the cocycle c must be bounded.*

Proof. Assume that $\beta_t \to \text{id}$ uniformly on the unit ball of pMp. Then $\beta_t \to \text{id}$ uniformly on the unit ball of Mz, where z is the central support of p in M. Therefore, we have that $\tau(\beta_t(u_g)u_g^* z) \to \tau(z)$, uniformly in $g \in \Gamma$. Since $E_M(\beta_t(u_g)) = \exp(-t^2 \|c(g)\|^2)u_g$ and since the conditional expectation E_M is trace-preserving, it follows that $\exp(-t^2 \|c(g)\|^2) \to 1$, uniformly in $g \in \Gamma$, and this implies that c is bounded. ☐

Lemma 4.7. *Assume that M is a II$_1$ factor such that $M^r \cong L\Lambda$ for some countable group Λ and $r > 0$. Let $\Delta : M \to M^r \mathbin{\overline{\otimes}} M$ be the amplified comultiplication defined in Section 2.7.*

If $id \otimes \beta_t \to id$ uniformly on the unit ball of $q\Delta(M)q$, for some non-zero projection $q \in \Delta(M)$, then the cocycle c must be bounded.

Proof. Let $q \in \Delta(M)$ and assume that $\mathrm{id} \otimes \beta_t \to \mathrm{id}$ uniformly on the unit ball of $q\Delta(M)q$. Then $\mathrm{id} \otimes \beta_t \to \mathrm{id}$ uniformly on the unit ball of $q\Delta(\mathrm{L}\Lambda)q$ and since $\{v_s\}_{s\in\Lambda}$ is a group of unitaries generating $\mathrm{L}\Lambda$, we have that $\mathrm{id}\otimes\beta_t \to \mathrm{id}$ uniformly on $\{q\Delta(v_s)q = q(v_s \otimes v_s)q \mid s \in \Lambda\}$. Hence there exists a non-zero projection $q_1 \in \mathrm{L}\Lambda$ such that $q \leq 1 \otimes q_1$ and $\beta_t \to \mathrm{id}$ uniformly on $\{q_1 v_s q_1 \mid v \in \Lambda\}$. Since $\{v_s\}_{s\in\Lambda}$ generate $\mathrm{L}\Lambda$, by Lemma 4.3, it follows that $\beta_t \to \mathrm{id}$ uniformly on the unit ball of $q_2(\mathrm{L}\Lambda)q_2$, where q_2 is the smallest projection in $\mathrm{L}\Lambda$ such that $q_1 \leq q_2$. Since q_1 is non-zero, it follows that q_2 is also non-zero and then, by Lemma 4.6, the cocycle c must be bounded. $\square$

Chapter 5

Normalizers of amenable subalgebras

One of the main technical tools in the proof of W*-superrigidity in [IPV10] is based on Popa's *clustering techniques* from [Po04]. Our approach in [BV12] and [Be14] does not use these techniques, but uses the recent dichotomy results for amenable subalgebras of [PV11], [PV12], [Io12b] and [Va13] instead. As a consequence, we can also prove W*-superrigidity for certain subgroups of generalized wreath products, see Theorem 1.2.2. In this chapter, we discuss briefly these dichotomy results and their applications and we prove results that allow us to have a good control of the normalizers of (relatively) amenable subalgebras.

5.1 Dichotomy for amenable subalgebras

Let (M, τ) be a tracial von Neumann algebra. Recall from [Oz03] that (M, τ) is called *solid* if the relative commutant $A' \cap M$ of any diffuse von Neumann subalgebra $A \subset M$ is amenable. Ozawa showed in [Oz03] that the group von Neumann algebra $\mathrm{L}\Gamma$ of any hyperbolic group Γ is solid. Moreover, (M, τ) is called *strongly solid* if the normalizer of any diffuse amenable von Neumann subalgebra of M is amenable. In [OP07] is shown that the free group factors $\mathrm{L}\mathbb{F}_n$ are strongly solid, while in [CS11] is shown that all group von Neumann algebras of hyperbolic groups are strongly solid.

Crossed products $M = B \rtimes \Gamma$ are typically not strongly solid, but for certain countable groups, one can prove some *relative strong solidity*, in the following sense: if $A \subset M$ is a von Neumann subalgebra that is amenable relative to B, then either $A \prec_M B$ or the normalizer of A in M is amenable relative to B. More precisely, we have the following result.

Theorem 5.1. *[PV11, Theorem 1.6] Let Γ be a weakly amenable group that admits a proper 1-cocycle into an orthogonal representation that is weakly contained in the left regular representation. Let $\Gamma \curvearrowright (B, \tau)$ be any trace-preserving action of Γ on a tracial von Neumann algebra (B, τ). Denote $M = B \rtimes \Gamma$ and let $A \subset M$ be a von Neumann subalgebra that is amenable relative to B. Then either $A \prec_M B$ or the normalizer $\mathcal{N}_M(A)''$ is amenable relative to B.*

This result has been used in [PV11, Theorem 1.5] to prove that whenever $\mathbb{F}_n \curvearrowright X$ and $\mathbb{F}_m \curvearrowright Y$ are two arbitrary free ergodic p.m.p. actions with $\mathrm{L}^\infty(X) \rtimes \mathbb{F}_n \cong \mathrm{L}^\infty(Y) \rtimes \mathbb{F}_m$, then $n = m$. In other words, this provides the first example of a countable group Γ with the property that any group measure space construction $\mathrm{L}^\infty(X) \rtimes \Gamma$, arising from an arbitrary free ergodic p.m.p. action $\Gamma \curvearrowright X$, has unique Cartan subalgebra, up to unitary conjugacy. Such a group Γ is called *Cartan-rigid* or shortly *$\mathcal{C}$-rigid*. By [PV11, Theorem 1.2], we also have that any weakly amenable group with positive first ℓ^2-Betti number is $\mathcal{C}$-rigid. It is conjectured in [PV11] that any countable group with at least one non-zero ℓ^2-Betti number is $\mathcal{C}$-rigid.

The same idea has been developed further on in [PV12], [Io12b] and [Va13] to obtain similar results for other classes of countable groups.

Theorem 5.2. *[PV12, Theorem 1.4] Let Γ be a weakly amenable, bi-exact group and let $\Gamma \curvearrowright (B, \tau)$ be any trace-preserving action of Γ on a tracial von Neumann algebra (B, τ). Denote $M = B \rtimes \Gamma$ and let $A \subset pMp$ be a von Neumann subalgebra that is amenable relative to B, where $p \in M$ is a non-zero projection. Then either $A \prec_M B$ or the normalizer $\mathcal{N}_{pMp}(A)''$ is amenable relative to B.*

In particular, any (non-amenable) weakly amenable, bi-exact countable group is $\mathcal{C}$-rigid. Examples of such groups are non-elementary hyperbolic groups, lattices in connected non-compact rank one simple Lie groups with finite center, the limit groups of Sela, etc.

Theorem 5.3. *[Io12b, Theorem 1.6] and [Va13, Theorems A and 4.1]*

Let M be either the amalgamated free product $M = M_1 *_B M_2$ of two tracial von Neumann algebras (M_i, τ) with common von Neumann subalgebra $B \subset M_i$ with respect to the unique trace-preserving conditional expectations or the HNN *extension $M = HNN(M_1, B, \theta)$ of the tracial von Neumann algebra (M_1, τ) with von Neumann subalgebra $B \subset M$ and trace-preserving embedding $\theta : B \to M_1$.*

Let $p \in M$ be a non-zero projection and $A \subset pMp$ be a von Neumann subalgebra that is amenable relative to M_1 inside M. Then at least one of the following statements holds:

- *$A \prec_M B$;*
- *$\mathcal{N}_{pMp}(A) \prec_M M_i$, for some $i = 1$ or 2;*
- *$\mathcal{N}_{pMp}(A)$ is amenable relative to B inside M.*

The case when M is an amalgamated free product was proven by Ioana in [Io12b, Theorem 1.6] under the additional assumption that the normalizer $\mathcal{N}_{pMp}(A)$ has spectral gap (see Section 3.2 for the definition). In particular, he proved in [Io12b, Theorem 7.1] that any non-degenerate amalgamated free product $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$, with Σ weakly malnormal in Γ, is $\mathcal{C}$-rigid. In [DI12, Corollary 1.7], a similar result was proven for HNN extensions, namely any non-degenerate HNN extension $\Gamma = \mathrm{HNN}(\Gamma_1, \Sigma, \theta)$, with Σ weakly malnormal in Γ, is $\mathcal{C}$-rigid. Later on, Vaes managed to remove the spectral gap assumption in [Va13, Theorem A] and to prove a similar result for HNN extensions in [Va13, Theorem 4.1].

5.2 Normalizers of amenable subalgebras

For the moment, we work in the following setup and under the following assumptions. We refer to Sections 2.1 and 2.5 for the definitions of weak amenability, class $\mathcal{S}$ and property Gamma.

Setup. We are given a II_1 factor M_0, a countable group Λ and a number $r > 0$ such that $M_0^r = L\Lambda$. We assume that $M_0 \subset M$, where M is of the form $M = B \rtimes \Gamma$ for a given trace-preserving action $\Gamma \curvearrowright (B, \tau)$ of a countable group Γ. We realize the amplification $(M_0 \mathbin{\overline{\otimes}} M_0)^r$ as $M_0^r \mathbin{\overline{\otimes}} M_0$ and we denote by $\Delta : M_0 \to M_0^r \mathbin{\overline{\otimes}} M_0$ the amplified comultiplication, as defined in Section 2.7.

Assumptions.

1. The group Γ satisfies one of the following conditions.

   (a) Γ is non-amenable, weakly amenable and belongs to class $\mathcal{S}$;

   (b) Γ is a non-degenerate amalgamated free product $\Gamma_1 *_\Sigma \Gamma_2$ or a non-degenerate HNN extension $\mathrm{HNN}(\Gamma_1, \Sigma, \theta)$ with Σ malnormal in Γ_1, respectively $\{\Sigma, \theta(\Sigma)\}$ malnormal in Γ_1;

   (c) Γ is non-amenable, weakly amenable and admits an unbounded 1-cocycle into a mixing representation that is weakly contained in the left regular representation of Γ.

2. We have $\Delta(M_0)' \cap M^r \mathbin{\overline{\otimes}} M = \mathbb{C}1$.

3. If $H < \Gamma$ is a subgroup of infinite index, we have that $M_0 \not\prec_M B \rtimes H$.

4. We have that M_0 is non-amenable relative to B inside M. Moreover, in the case 1.(b), we also assume that M_0 is non-amenable relative to $B \rtimes \Sigma$.

5. In the case 1.(c), if $(\beta_t)_{t\in\mathbb{R}}$ denotes the Gaussian deformation defined in Section 4.3, then $\mathrm{id} \otimes \beta_t$ does not converge to id uniformly on the unit ball of $q\Delta(M_0)q$, for any projection $q \in \Delta(M_0)$.

At a first reading, one may very well assume that $M_0 = M$. In that case, assumption 2 follows because Λ is an i.c.c. group, assumption 3 is trivially satisfied, assumption 4 is a consequence of Lemma 2.12 and Lemma 2.25 and assumption 5 follows from Lemma 4.7. This will be enough to prove Theorem 1.2.1. The general situation is only needed to prove Theorem 1.2.2.

The following theorem is a direct consequence of the main results in [PV11], [PV12] and [Va13].

Theorem 5.4. *Assume that we are in the setup and under the assumptions described above. If $Q \subset M^r \mathbin{\overline{\otimes}} M$ is a von Neumann subalgebra such that $\Delta(M_0) \subset \mathcal{N}_{M^r \overline{\otimes} M}(Q)''$ and such that Q is amenable relative to $M^r \mathbin{\overline{\otimes}} B$, then $Q \prec^f M^r \mathbin{\overline{\otimes}} B$.*

Proof. Throughout the proof, we view $M^r \mathbin{\overline{\otimes}} M$ as the crossed product $(M^r \mathbin{\overline{\otimes}} B) \rtimes \Gamma$. By assumption 2, we have that $\Delta(M_0)' \cap (M \mathbin{\overline{\otimes}} M)^r = \mathbb{C}1$. Since $\Delta(M_0) \subset \mathcal{N}_{M^r \overline{\otimes} M}(Q)''$, by Lemma 2.7.(a), it suffices to prove that $Q \prec M^r \mathbin{\overline{\otimes}} B$.

First assume that Γ satisfies assumption 1.(a). By [PV12, Theorem 1.4], we have that either $\Delta(M_0)$ is amenable relative to $M^r \mathbin{\overline{\otimes}} B$, or that $Q \prec M^r \mathbin{\overline{\otimes}} B$. Using Proposition 2.21.(d) and assumption 4, we see that the first option is impossible. So we indeed get that $Q \prec M^r \mathbin{\overline{\otimes}} B$.

Next assume that Γ satisfies assumption 1.(b). Denote $P := \mathcal{N}_{M^r \overline{\otimes} M}(Q)''$ and $\mathcal{B} := M^r \mathbin{\overline{\otimes}} B$. Suppose first that $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ is non-degenerate and Σ is malnormal in Γ_1 and notice that Σ is relatively malnormal in Γ by Theorem 2.26. Remark also that we can write $M^r \mathbin{\overline{\otimes}} M$ as an amalgamated free product

$$M^r \mathbin{\overline{\otimes}} M = (\mathcal{B} \rtimes \Gamma_1) *_{\mathcal{B}\rtimes\Sigma} (\mathcal{B} \rtimes \Gamma_2).$$

By Theorem 5.3, at least one of the following statements is true:

- $Q \prec \mathcal{B} \rtimes \Sigma$;
- $P \prec \mathcal{B} \rtimes \Gamma_i$, for some $i = 1$ or 2;
- P is amenable relative to $\mathcal{B} \rtimes \Sigma$.

If $Q \prec \mathcal{B} \rtimes \Sigma$, then we get that $Q \prec \mathcal{B}$. Indeed, since Σ is relatively malnormal in Γ there is an infinite index subgroup $\Lambda < \Gamma$ such that $\Sigma \cap g\Sigma g^{-1}$ is finite, for all $g \in \Gamma \setminus \Lambda$. Assume, by contradiction, that $Q \not\prec \mathcal{B}$. Then, by [Va10b, Lemma 6.4], it follows that $P \prec \mathcal{B} \rtimes \Lambda$, and hence $\Delta(M_0) \prec \mathcal{B} \rtimes \Lambda$, which, by Proposition 2.21.(d) and assumption 3, is not possible since Λ has infinite index in Γ. Thus we get that $Q \prec M^r \mathbin{\overline{\otimes}} B$.

If $P \prec \mathcal{B} \rtimes \Gamma_i$, for some $i = 1$ or 2, then $\Delta(M_0) \prec \mathcal{B} \rtimes \Gamma_i$, which contradicts Proposition 2.21.(c) and assumption 3, since Γ_i has infinite index in Γ, for all $i = 1, 2$.

If P is amenable relative to $\mathcal{B} \rtimes \Sigma$, for some $i = 1$ or 2, then $\Delta(M_0)$ is amenable relative to $\mathcal{B} \rtimes \Sigma$. By Proposition 2.21.(d) it follows that M_0 is amenable relative to $B \rtimes \Sigma$, but this contradicts assumption 4.

Assume now that $\Gamma = \mathrm{HNN}(\Gamma_1, \Sigma, \theta) = \langle \Gamma_1, t \mid t\Sigma t^{-1} = \theta(\Sigma) \rangle$ is non-degenerate and $\{\Sigma, \theta(\Sigma)\}$ is malnormal in Γ_1. By Theorem 2.27 we have that Σ is relatively malnormal in Γ. Using the construction in [FV10, Section 3], we can write $M^r \mathbin{\overline{\otimes}} M$ as an HNN extension $\mathrm{HNN}(\mathcal{B} \rtimes \Gamma_1, \mathcal{B} \rtimes \Sigma, \Theta)$, and hence, by Theorem 5.3, at least one of the following statements is true:

- $Q \prec \mathcal{B} \rtimes \Sigma$;
- $P \prec \mathcal{B} \rtimes \Gamma_1$;
- P is amenable relative to $\mathcal{B} \rtimes \Sigma$.

The last two alternatives cannot hold, as in the previous case, thus we have $Q \prec \mathcal{B} \rtimes \Sigma$, which implies that $Q \prec M^r \mathbin{\overline{\otimes}} B$, since $\Sigma < \Gamma$ is relatively malnormal.

Finally, assume that Γ satisfies assumption 1.(c). Let $c : \Gamma \to \mathcal{K}_{\mathbb{R}}$ be an unbounded 1-cocycle into the mixing orthogonal representation $(\pi, \mathcal{K}_{\mathbb{R}})$ of Γ which is weakly contained into the left regular representation λ of Γ.

Denote $P := \mathcal{N}_{M^r \overline{\otimes} M}(Q)''$ and write $\mathcal{M} := M^r \overline{\otimes} M$, $\mathcal{B} := M^r \overline{\otimes} B$, so that $\mathcal{M} \cong \mathcal{B} \rtimes \Gamma$. Denote by $\mathcal{K}^\pi$ the $\mathcal{M}$-$\mathcal{M}$-bimodule associated to π and by $(\varphi_t)_{t\geq 0}$ the group of unital normal completely positive maps associated to the 1-cocycle c. Assume, by contradiction, that $Q \nprec M^r \overline{\otimes} B$. By [PV11, Theorem 3.1], at least one of the following statements must be true:

1. The $\mathcal{M}$-$\mathcal{M}$-bimodule $\mathcal{K}^\pi$ is left P-amenable;
2. There exist $t, \delta > 0$ such that $\|\varphi_t(a)\|_2 \geq \delta$, for all $a \in \mathcal{U}(Q)$.

Case 1. If ${}_{\mathcal{M}}\mathcal{K}^\pi{}_{\mathcal{M}}$ is left P-amenable, then ${}_{\mathcal{M}}\mathcal{K}^\pi{}_{\mathcal{M}}$ is left $\Delta(M_0)$-amenable. Since π is weakly contained in the left regular representation, it follows that ${}_{\mathcal{M}}\mathcal{K}^\pi{}_{\mathcal{M}} \prec {}_{\mathcal{M}}(L^2(\mathcal{M}) \otimes_{\mathcal{B}} L^2(\mathcal{M}))_{\mathcal{M}}$, and therefore, by [PV11, Corollary 2.5], we get that

$${}_{\mathcal{M}}(L^2(\mathcal{M}) \otimes_{\mathcal{B}} L^2(\mathcal{M}))_{\mathcal{M}} \text{ is left } \Delta(M_0)\text{-amenable.}$$

By [PV11, Proposition 2.4] this further implies that ${}_{\mathcal{M}}L^2(\mathcal{M})_{\mathcal{B}}$ is left $\Delta(M_0)$-amenable, i.e. $\Delta(M_0)$ is amenable relative to $M^r \overline{\otimes} B$. Finally, by Proposition 2.21.(d), we get that M_0 is amenable relative to B, which contradicts assumption 4.

Case 2. Assume that there exist $t, \delta > 0$ such that $\|\varphi_t(a)\|_2 \geq \delta$, for all $a \in \mathcal{U}(Q)$. Let $(\beta_t)_{t\in\mathbb{R}} \in Aut(\widetilde{M})$ be the Gaussian deformation on M, defined in Section 4.3.

Since π is mixing, by [Va10b, Proposition 3.9], there is a non-zero projection $p \in Z(P)$ such that

$$\mathrm{id} \otimes \beta_t \to \mathrm{id} \text{ uniformly on the unit ball of } Qp.$$

Now, since moreover $Q \nprec \mathcal{B}$, it follows by Theorem 4.4 that

$$\mathrm{id} \otimes \beta_t \to \mathrm{id} \text{ uniformly on the unit ball of } Pq,$$

where $q \in Z(P)$ is the smallest projection such that $p \leq q$. In particular, q is non-zero and since $\Delta(M_0) \subset P$ we get that $\mathrm{id} \otimes \beta_t \to \mathrm{id}$ uniformly on the unit ball of $\Delta(M_0)q$, but this contradicts assumption 5.

□

We also need the next result, which is an analogue of [Io12a, Corollary 2.12]. Since the first part of the proof goes exactly as in Ioana's proof, we are rather brief, pointing out the arguments that are different.

Theorem 5.5. *Let Γ be a non-amenable group that admits an unbounded 1-cocycle c into the left regular representation. Let $\Gamma \curvearrowright (B, \tau)$ be a trace-preserving action and denote $M := B \rtimes \Gamma$. Let $\Sigma < \Gamma$ be a subgroup and assume that the cocycle c is bounded on Σ. Denote $M_1 = B \rtimes \Sigma$ and let $Q \subset pMp$ be a von Neumann subalgebra that is amenable relative to M_1, for some non-zero projection $p \in M$. Denote $P = \mathcal{N}_{pMp}(Q)''$. Consider the Gaussian deformation $(\beta_t)_{t\in\mathbb{R}} \in Aut(\widetilde{M})$ defined in Section 4.3. Then at least one of the following statements holds:*

- *There is a non-zero projection $q \in Q' \cap pMp$ such that Qq is amenable relative to B;*
- *There is a non-zero projection $r \in Z(P)$ such that $\beta_t \to id$ uniformly on the unit ball of Pr.*

Proof. We may assume that the cocycle c is zero on Σ. Since Q is amenable relative to M_1 inside M, there exists a net $(\xi_i)_{i\in I} \in L^2(p\langle M, e_{M_1}\rangle p)$ such that

$$\lim_{i\in I} \|a\xi_i - \xi_i a\|_2 = 0, \quad \text{for all } a \in Q, \tag{5.1}$$

and

$$\lim_{i\in I} \langle x\xi_i, \xi_i\rangle = \lim_{i\in I} \langle \xi_i x, \xi_i\rangle = \tau(x), \quad \text{for all } x \in pMp. \tag{5.2}$$

Since c is zero on Σ, then β_t is identity on $M_1 = B \rtimes \Sigma$ and hence, we can extend β_t to a trace-preserving automorphism β_t of the basic construction $\langle \widetilde{M}, e_{M_1}\rangle$, by letting $\beta_t(e_{M_1}) = e_{M_1}$.

Denote by $\mathcal{H}$ the L^2-closed linear span of the set

$$Me_{M_1}\widetilde{M} := \{xe_{M_1}y \ ; \ x \in M, \ y \in \widetilde{M}\}$$

and let $e_{\mathcal{H}}$ be the orthogonal projection of $\mathrm{L}^2(\langle \widetilde{M}, e_{M_1}\rangle)$ onto $\mathcal{H}$.

Fix $t \in \mathbb{R}$. Since, by construction, one can see $\mathrm{L}^2(\langle M, e_{M_1}\rangle)$ as a subspace of $\mathrm{L}^2(\langle \widetilde{M}, e_{M_1}\rangle)$, we may define the net $(\xi_i^t)_{i\in I} \subset \mathrm{L}^2(\langle \widetilde{M}, e_{M_1}\rangle)$ by letting $\xi_i^t := \beta_t(\xi_i)$, for all $i \in I$. We prove now that the following relations hold:

$$\lim_{i\in I} \left\|x\xi_i^t\right\|_2 \le \|x\|_2 \text{ and } \lim_{i\in I} \left\|\xi_i^t x\right\|_2 \le \|x\|_2\,, \tag{5.3}$$

$$\limsup_{i\in I} \left\|xe_{\mathcal{H}}(\xi_i^t)\right\|_2 \le \|x\|_2 \tag{5.4}$$

and

$$\limsup_{i\in I} \left\|a\xi_i^t - \xi_i^t a\right\|_2 \le 2\,\|a - \beta_t(a)\|_2\,, \tag{5.5}$$

for every $a \in Q$ and for every $x \in \widetilde{M}$.

Indeed, since β_t is trace-preserving, $\xi_i \in p\mathcal{H}$ and $(\widetilde{M} \ominus M)\mathcal{H} \perp \mathcal{H}$, by using the first part of (5.2), we get that

$$\begin{aligned}
\lim_{i\in I} \left\|x\xi_i^t\right\|_2^2 &= \lim_{i\in I} \langle x\beta_t(\xi_i), x\beta_t(\xi_i)\rangle \\
&= \lim_{i\in I} \left\langle \beta_t^{-1}(x^*x)\xi_i, \xi_i\right\rangle \\
&= \lim_{i\in I} \left\langle pE_M(\beta_t^{-1}(x^*x))p\xi_i, \xi_i\right\rangle \\
&= \tau(pE_M(\beta_t^{-1}(x^*x))p) \\
&= \tau(x^*x\beta_t(p)) \leq \|x\|_2^2 .
\end{aligned}$$

The second inequality of (5.3) follows similarly using the second part of the equation (5.2).

Now, since $(\widetilde{M} \ominus M)\mathcal{H} \perp \mathcal{H}$ and $\mathcal{H}$ is a left M-module, it follows that

$$\begin{aligned}
\|xe_{\mathcal{H}}(\xi_i^t)\|_2^2 &= \langle xe_{\mathcal{H}}(\xi_i^t), xe_{\mathcal{H}}(\xi_i^t)\rangle \\
&= \langle E_M(x^*x)e_{\mathcal{H}}(\xi_i^t), e_{\mathcal{H}}(\xi_i^t)\rangle \\
&= \left\langle e_{\mathcal{H}}(E_M(x^*x)^{1/2}\xi_i^t), e_{\mathcal{H}}(E_M(x^*x)^{1/2}\xi_i^t)\right\rangle \\
&= \left\|e_{\mathcal{H}}(E_M(x^*x)^{1/2}\xi_i^t)\right\|_2^2 \\
&\leq \left\|E_M(x^*x)^{1/2}\xi_i^t\right\|_2^2 ,
\end{aligned}$$

and hence, passing to $\limsup$ and using (5.3), we get that

$$\limsup_{i\in I} \left\|xe_{\mathcal{H}}(\xi_i^t)\right\|_2 \leq \left\|E_M(x^*x)^{1/2}\right\|_2 = \|x\|_2 .$$

Finally, to prove (5.5), we have that

$$\left\|a\xi_i^t - \xi_i^t a\right\|_2 \leq \left\|(a-\beta_t(a))\xi_i^t\right\|_2 + \left\|\xi_i^t(a-\beta_t(a))\right\|_2 + \|a\xi_i - \xi_i a\|_2 .$$

Passing to $\limsup$ and using (5.3) and (5.1), we get that

$$\limsup_{i\in I} \left\|a\xi_i^t - \xi_i^t a\right\|_2 \leq 2\,\|a-\beta_t(a)\|_2.$$

For any $t > 0$, consider the net $\eta_i^t := \xi_i^t - e_{\mathcal{H}}(\xi_i^t)$ and denote $\delta_i^t := \|\eta_i^t\|_2$. We have now two different cases which will be treated separately.

Case 1. Assume that there exists a $t > 0$ such that $\limsup_{i\in I} \delta_i^t < \frac{5\,\|p\|_2}{11}$.

Fix $a \in \mathcal{U}(Q)$. Since $(\widetilde{M} \ominus M)\mathcal{H} \perp \mathcal{H}$ and $\mathcal{H}$ is a left M-module, it follows that

$$\begin{aligned}\|E_M(\beta_t(a))\xi_i^t\|_2 &\geq \|e_{\mathcal{H}}(E_M(\beta_t(a))\xi_i^t)\|_2\\ &= \|e_{\mathcal{H}}(\beta_t(a)e_{\mathcal{H}}(\xi_i^t))\|_2\\ &\geq \|e_{\mathcal{H}}(\beta_t(a)\xi_i^t)\|_2 - \delta_i^t\\ &\geq \|e_{\mathcal{H}}(\xi_i^t\beta_t(a))\|_2 - \|a\xi_i - \xi_i a\|_2 - \delta_i^t.\end{aligned}$$

On the other hand, since β_t is trace-preserving and $\mathcal{H}$ is also a right $\widetilde{M}$-module, we have that

$$\left\|e_{\mathcal{H}}(\xi_i^t\beta_t(a))\right\|_2 = \left\|e_{\mathcal{H}}(\xi_i^t)\beta_t(a)\right\|_2 \geq \left\|\xi_i^t\beta_t(a)\right\|_2 - \delta_i^t = \|\xi_i a\|_2 - \delta_i^t.$$

Thus

$$\left\|E_M(\beta_t(a))\xi_i^t\right\|_2 \geq \|\xi_i a\|_2 - \|a\xi_i - \xi_i a\|_2 - 2\delta_i^t,$$

and hence, by (5.1), (5.2) and (5.3), we get that

$$\begin{aligned}\|E_M(\beta_t(a))\|_2 &\geq \lim_{i\in I}\left\|E_M(\beta_t(a))\xi_i^t\right\|_2\\ &\geq \liminf_{i\in I}\left(\|\xi_i a\|_2 - \|a\xi_i - \xi_i a\|_2 - 2\delta_i^t\right)\\ &= \|a\|_2 - 2\limsup_{i\in I}\delta_i^t\\ &= \|p\|_2 - 2\limsup_{i\in I}\delta_i^t > \frac{\|p\|_2}{11}.\end{aligned}$$

Therefore, for all $a \in \mathcal{U}(Q)$, we have that

$$\|E_M(\beta_t(a))\|_2 > \frac{\|p\|_2}{11},$$

and hence, by [Va10b, Proposition 3.9], there exists a non-zero projection $q_0 \in Z(P)$ such that

$$\beta_t \to \mathrm{id} \text{ uniformly on the unit ball of } Qq_0. \tag{5.6}$$

Furthermore, by (5.6) and Theorem 4.4, it follows that

- either $Q \prec_M B$,
- or $\beta_t \to \mathrm{id}$ uniformly on the unit ball of Pr, where $r \in Z(P)$ is the smallest projection such that $q_0 \leq r$.

Note that, by [Io12b, Remark 2.2], the first alternative yields a non-zero projection $q \in Q' \cap pMp$ such that Qq is amenable relative to B, so the proof in Case 1 is done.

Case 2. Suppose that, for all $t > 0$, we have $\limsup_{i \in I} \delta_i^t \geq \frac{5\|p\|_2}{11}$.

In this case we prove that there exists a net $(\eta_j)_{j \in J} \subset \mathrm{L}^2(\langle \widetilde{M}, e_{M_1}\rangle) \ominus \mathcal{H}$ that satisfies the following three conditions:

$$\limsup_{j \in J} \|p\eta_j\|_2 > 0, \tag{5.7}$$

$$\limsup_{j \in J} \|x\eta_j\|_2 \leq 2\|x\|_2, \text{ for all } x \in pMp, \tag{5.8}$$

and

$$\lim_{j \in J} \|a\eta_j - \eta_j a\|_2 = 0, \text{ for all } a \in Q. \tag{5.9}$$

Let J denote the set of triples $j := (X, Y, \varepsilon)$ consisting of finite subsets $X \subset Q$, $Y \subset pMp$ and $\varepsilon > 0$. Fix such a triple $j = (X, Y, \varepsilon)$. Since β_t converges to identity, L^2-pointwise on M, we can find a $t > 0$ such that, for all $a \in Q$, we have

$$\|a - \beta_t(a)\|_2 < \varepsilon/2 \text{ and } \|p - \beta_t(p)\|_2 < \|p\|_2/10. \tag{5.10}$$

Let $a \in X$ and $x \in Y$. Since $\eta_i^t = (1 - e_{\mathcal{H}})\xi_i^t$ and $a \in Q$, we get by (5.4) that

$$\left\|a\eta_i^t - \eta_i^t a\right\|_2 \leq \left\|a\xi_i^t - \xi_i^t a\right\|_2,$$

and passing to lim sup and using (5.5) and (5.10), it follows that

$$\limsup_{i \in I} \left\|a\eta_i^t - \eta_i^t a\right\|_2 < \varepsilon. \tag{5.11}$$

Moreover, by (5.3) and (5.4), we have that

$$\limsup_{i \in I} \left\|x\eta_i^t\right\|_2 \leq 2\|x\|_2, \tag{5.12}$$

and by (5.3), (5.2) and (5.10), we also get that

$$\begin{aligned}\limsup_{i\in I}\left\|p\eta_i^t\right\|_2 &\geq \limsup_{i\in I}\left(\left\|p\xi_i^t\right\|_2-\left\|e_{\mathcal{H}}(\xi_i^t)\right\|_2\right)\\ &= \|p\beta_t(p)\|_2-\liminf_{i\in I}\left\|e_{\mathcal{H}}(\xi_i^t)\right\|_2\\ &\geq \|p\beta_t(p)\|_2-\left(\|p\|_2^2-\limsup_{i\in I}\left\|\eta_i^t\right\|_2^2\right)^{1/2} \\ &> \left(\frac{9}{10}-\frac{4\sqrt{6}}{11}\right)\|p\|_2>0.\end{aligned} \tag{5.13}$$

Combining (5.11), (5.12) and (5.13) it follows that, for some $i\in I$, the vectors $\eta_j := \eta_i^t$ satisfy the required conditions (5.7), (5.8) and (5.9).

Thus, by Lemma 2.18, there exists a non-zero projection $q\in Q'\cap pMp$ such that the qMq-M-bimodule

$$q\mathrm{L}^2(\langle\widetilde{M},e_{M_1}\rangle)\ominus\mathcal{H} \text{ is left } Qq\text{-amenable.}$$

By the definition of $\mathcal{H}$ we have that, as M-M-bimodules,

$$\mathrm{L}^2(\langle\widetilde{M},e_{M_1}\rangle)\ominus\mathcal{H}\cong\mathrm{L}^2(\widetilde{M}\ominus M)\otimes_{M_1}\mathrm{L}^2(\widetilde{M}),$$

so, it follows that $q\mathrm{L}^2(\widetilde{M}\ominus M)\otimes_{M_1}\mathrm{L}^2(\widetilde{M})$ is left Qq-amenable.

By [PV11, Proposition 2.4], it follows that the qMq-M_1-bimodule $q\mathrm{L}^2(\widetilde{M}\ominus M)$ is left Qq-amenable. Since $\mathrm{L}^2(\widetilde{M}\ominus M)$ is weakly contained in $\mathrm{L}^2(M)\otimes_B\mathrm{L}^2(M)$ (see for instance [Va10b, Lemma 3.5]), then by [PV11, Corollary 2.5] and [PV11, Proposition 2.4], we get that the qMq-B-bimodule $q\mathrm{L}^2(M)$ is left Qq-amenable. Thus Qq is amenable relative to B, for some non-zero projection $q\in Q'\cap pMp$, and this concludes the proof of Case 2. □

Chapter 6

Comultiplications and relative commutants

In this chapter we continue our study of the amplified comultiplication defined in Section 2.7 and we prove some technical results needed in the proof of the main theorem. We start with the following elementary lemma.

Lemma 6.1. *Let G and $\mathcal{G}$ be countable groups and $\gamma_i : G \to \mathcal{G}$ group homomorphisms, with $i = 1, 2$. Assume that for every $h \in \mathcal{G} - \{e\}$, the set $\{\gamma_1(g)h\gamma_1(g)^{-1} \mid g \in G\}$ is infinite. Then the following statements are equivalent.*

(a) There exists an $h \in \mathcal{G}$ such that $\gamma_1(g) = h\gamma_2(g)h^{-1}$ for all $g \in G$.

(b) There exists a finite subset $\mathcal{F} \subset \mathcal{G}$ such that $\mathcal{F} \cap \gamma_1(g)\mathcal{F}\gamma_2(g)^{-1} \neq \emptyset$ for all $g \in G$.

(c) The unitary representation

$$\pi : G \to \mathcal{U}(\ell^2\mathcal{G}) : \pi(g)\xi = u_{\gamma_1(g)}\xi u^*_{\gamma_2(g)}$$

is not weakly mixing.

Proof. The equivalence of (b) and (c) follows from Lemma 2.20. The implication (a) $\Rightarrow$ (b) is trivial by taking $\mathcal{F} = \{h\}$. Conversely assume that (b) holds. By Lemma 2.20, we can take an $h \in \mathcal{G}$ such that $\mathcal{F}_1 := \{\gamma_1(g)h\gamma_2(g)^{-1} \mid g \in G\}$ is a finite set. It follows that $\mathcal{F}_1\mathcal{F}_1^{-1}$ is a finite subset of $\mathcal{G}$ that is globally invariant under $(\operatorname{Ad}\gamma_1(g))_{g\in G}$. By our assumptions, it follows that

$\mathcal{F}_1\mathcal{F}_1^{-1} = \{e\}$. This means that $\mathcal{F}_1$ is a singleton. So $\mathcal{F}_1 = \{h\}$ and we conclude that $\gamma_1(g) = h\gamma_2(g)h^{-1}$ for all $g \in G$. $\square$

Lemma 6.2. *Let Λ be an i.c.c. group and $\alpha, \beta \in \mathrm{Aut}(L\Lambda)$. Denote by $(v_s)_{s\in\Lambda}$ the canonical group of unitaries generating $L\Lambda$. Let $\Delta : L\Lambda \to L\Lambda \mathbin{\overline{\otimes}} L\Lambda$ be the comultiplication defined by $\Delta(v_s) = v_s \otimes v_s$. If $(\alpha \otimes \beta)\Delta(L\Lambda) \prec \Delta(L\Lambda)$, there exist unitaries $V, W \in L\Lambda$, characters $\omega, \mu : \Lambda \to \mathbb{T}$ and an automorphism $\delta \in \mathrm{Aut}(\Lambda)$ such that $\alpha(v_s) = \omega(s)\, V v_{\delta(s)} V^*$ and $\beta(v_s) = \mu(s)\, W v_{\delta(s)} W^*$ for all $s \in \Lambda$.*

Proof. We start by proving the following claim: if Λ admits a sequence of elements $s_n \in \Lambda$ such that

$$\lim_n \|E_{\Delta(\mathrm{L}\Lambda)}(v_x^*\alpha(v_{s_n}) \otimes \beta(v_{s_n})v_y^*)\|_2 = 0 \quad \text{for all } \ x, y \in \Lambda\ , \tag{6.1}$$

then $(\alpha \otimes \beta)\Delta(\mathrm{L}\Lambda) \not\prec \Delta(\mathrm{L}\Lambda)$. Indeed, if (6.1) holds, we multiply from the left and right by elements of the form $\Delta(v_a)$, $\Delta(v_b)$ and conclude that for all $x, y, a, b \in \Lambda$

$$\lim_n \|E_{\Delta(\mathrm{L}\Lambda)}\big((v_x \otimes v_y)\,(\alpha \otimes \beta)\Delta(v_{s_n})\,(v_a \otimes v_b)\big)\|_2 = 0.$$

Using $\|\cdot\|_2$-approximations, it follows that the same holds when we replace $v_x \otimes v_y$ and $v_a \otimes v_b$ by arbitrary elements of $\mathrm{L}\Lambda \mathbin{\overline{\otimes}} \mathrm{L}\Lambda$. This then means that $(\alpha \otimes \beta)\Delta(\mathrm{L}\Lambda) \not\prec \Delta(\mathrm{L}\Lambda)$ and hence this proves the claim.

Our assumption is that $(\alpha \otimes \beta)\Delta(\mathrm{L}\Lambda) \prec \Delta(\mathrm{L}\Lambda)$. So by the claim above, there is no sequence of elements $s_n \in \Lambda$ satisfying (6.1). This means that there are finitely many $x_i, y_i \in \Lambda$, with $i = 1, \ldots, k$, and a $\delta > 0$ such that

$$\sum_{i=1}^k \|E_{\Delta(\mathrm{L}\Lambda)}(v_{x_i}^*\alpha(v_s) \otimes \beta(v_s)v_{y_i}^*)\|_2^2 \geq \delta \quad \text{for all } \ s \in \Lambda\ .$$

The left hand side can be computed and we conclude that

$$\sum_{i=1}^k \sum_{t\in\Lambda} |\tau(v_{x_it}^*\alpha(v_s))|^2\, |\tau(v_{ty_i}^*\beta(v_s))|^2 \geq \delta \quad \text{for all } \ s \in \Lambda\ . \tag{6.2}$$

As in [IPV10, Formula (3.1)], we define the *height* of an element $a \in \mathrm{L}\Lambda$ as

$$h_\Lambda(a) := \max\{|\tau(v_t^* a)| \mid t \in \Lambda\}\ .$$

Using (6.2), we find that for all $s \in \Lambda$, we have

$$\begin{aligned}\delta &\leq \sum_{i=1}^{k} \sum_{t \in \Lambda} |\tau(v_{x_i t}^* \alpha(v_s))|^2 \, |\tau(v_{t y_i}^* \beta(v_s))|^2 \\ &\leq h_\Lambda(\alpha(v_s))^2 \; \sum_{i=1}^{k} \sum_{t \in \Lambda} |\tau(v_{t y_i}^* \beta(v_s))|^2 \\ &= k \, h_\Lambda(\alpha(v_s))^2 \; .\end{aligned}$$

So we get that $h_\Lambda(\alpha(v_s)) \geq \sqrt{\delta/k}$ for all $s \in \Lambda$. It then follows from [IPV10, Theorem 3.1] that there exist a unitary $V \in \mathrm{L}\Lambda$, a character $\omega : \Lambda \to \mathbb{T}$ and an automorphism $\delta_1 \in \mathrm{Aut}(\Lambda)$ such that $\alpha(v_s) = \omega(s) \, V v_{\delta_1(s)} V^*$ for all $s \in \Lambda$.

By symmetry, we find the same description of the automorphism β, yielding a unitary $W \in \mathrm{L}\Lambda$, a character $\mu : \Lambda \to \mathbb{T}$ and an automorphism $\delta_2 \in \mathrm{Aut}(\Lambda)$ such that $\beta(v_s) = \mu(s) \, W v_{\delta_2(s)} W^*$ for all $s \in \Lambda$. It remains to prove that up to an inner conjugacy, $\delta_1 = \delta_2$. Replacing α by $(\mathrm{Ad}\, V^*) \circ \alpha$ and replacing β by $(\mathrm{Ad}\, W^*) \circ \beta$, we still have that $(\alpha \otimes \beta)\Delta(\mathrm{L}\Lambda) \prec \Delta(\mathrm{L}\Lambda)$. So there exist finitely many $x_i, y_i \in \Lambda$, with $i = 1, \ldots, k$, and a $\delta > 0$ such that (6.2) holds. Since now $\alpha(v_s) = \omega(s) \, v_{\delta_1(s)}$ and $\beta(v_s) = \mu(s) \, v_{\delta_2(s)}$, the left hand side of (6.2) is zero, unless there exists an $i \in \{1, \ldots, k\}$ and a $t \in \Lambda$ satisfying $\delta_1(s) = x_i t$ and $\delta_2(s) = t y_i$. This means that for every $s \in \Lambda$, there exists an $i \in \{1, \ldots, k\}$ such that $\delta_1(s) y_i \delta_2(s)^{-1} = x_i$. Since Λ is i.c.c., it then follows from Lemma 6.1 that δ_1 and δ_2 are equal up to inner conjugacy. ☐

Let Λ be an i.c.c. group and assume that $\mathrm{L}\Lambda$ does not have property Gamma, so that $\mathrm{Out}(\mathrm{L}\Lambda)$ is a Polish group (see Section 2.1 for notations and terminology). For every character $\omega \in \widehat{\Lambda}$, we denote by α_ω the automorphism of $\mathrm{L}\Lambda$ given by $\alpha_\omega(v_s) = \omega(s) \, v_s$ for all $s \in \Lambda$. Using the i.c.c. property, one checks that the map $\omega \mapsto \alpha_\omega$ embeds $\widehat{\Lambda}$ continuously into $\mathrm{Out}(\mathrm{L}\Lambda)$. Since $\widehat{\Lambda}$ is compact, we can thus view $\widehat{\Lambda}$ as a compact subgroup of $\mathrm{Out}(\mathrm{L}\Lambda)$.

A countable subgroup $\mathcal{A}$ of a Polish group $\mathcal{B}$ is said to be *discrete* if there exists a neighborhood $\mathcal{U}$ of the identity e in $\mathcal{B}$ such that $\mathcal{U} \cap \mathcal{A} = \{e\}$.

Lemma 6.3. *Let M_0 be a II$_1$ factor without property Gamma. Let $r > 0$ and $M_0^r = L\Lambda$ for some countable group Λ. Denote by $\Delta : M_0 \to (M_0 \mathbin{\overline{\otimes}} M_0)^r$ the amplified comultiplication as in Section 2.7.*

Assume that (M, τ) is a tracial von Neumann algebra with $M_0 \subset M$ and $M_0' \cap M = \mathbb{C}1$. Let $\mathcal{L} \subset \mathcal{N}_M(M_0)$ be a subgroup such that $M = (M_0 \cup \mathcal{L})''$.

Finally assume that the image of $\mathcal{L}$ in $\mathrm{Out}(M_0)$ *is a discrete torsion-free subgroup. Then the following holds.*

(a) If $\mathcal{H} \subset L^2((M \mathbin{\overline{\otimes}} M)^r) \ominus L^2(\Delta(M_0))$ *is a non-zero* $\Delta(M_0)$*-*$\Delta(M_0)$*-subbimodule of finite left* $\Delta(M_0)$*-dimension, then there exist automorphisms* $\beta_1, \ldots, \beta_k \in \mathrm{Aut}(M_0)$ *and a unitary*

$$\psi : \mathcal{H} \to L^2(M_0)^{\oplus k} : \xi \mapsto (\psi_1(\xi), \ldots, \psi_k(x))$$

such that for all $x, y \in M_0, \xi \in \mathcal{H}, i = 1, \ldots, k$

$$\psi_i(\Delta(x)\, \xi\, \Delta(y)) = x\, \psi_i(\xi)\, \beta_i(y)$$

and such that every β_i *generates a discrete infinite subgroup of* $\mathrm{Out}(M_0)$*.*

(b) We have $\Delta(M_0)' \cap (M \mathbin{\overline{\otimes}} M)^r = \mathbb{C}1$*.*

Proof. First note that statement (b) is a consequence of statement (a). Take an element T in $\Delta(M_0)' \cap (M \mathbin{\overline{\otimes}} M)^r$ and write $S := T - E_{\Delta(M_0)}(T)$. Since M_0 is a factor, it suffices to prove that $S = 0$. So assume that $S \neq 0$. Denote by $\mathcal{H}$ the closure of $\Delta(M_0)S$. Then $\mathcal{H}$ is a $\Delta(M_0)$-$\Delta(M_0)$-subbimodule of $L^2((M \mathbin{\overline{\otimes}} M)^r) \ominus L^2(\Delta(M_0))$ that has finite left dimension. By construction $\mathcal{H}$ contains the non-zero vector S satisfying $\Delta(x)S = S\Delta(x)$ for all $x \in M_0$. Write $\mathcal{H}$ as in (a). Since all automorphisms β_i are outer, we have that $\psi_i(S) = 0$ for all $i \in \{1, \ldots, k\}$. So $S = 0$, contradicting our assumption.

We now start proving statement (a). Take a projection $p \in \mathrm{M}_n(\mathbb{C}) \otimes M_0$ with $(\mathrm{Tr} \otimes \tau)(p) = r$. Realize $M_0^r := p(\mathrm{M}_n(\mathbb{C}) \otimes M_0)p$ and $(M_0 \mathbin{\overline{\otimes}} M_0)^r = M_0^r \mathbin{\overline{\otimes}} M_0$. Denote by $\Delta : \mathrm{L}\Lambda \to \mathrm{L}\Lambda \mathbin{\overline{\otimes}} \mathrm{L}\Lambda : \Delta(v_s) = v_s \otimes v_s$ the original comultiplication. During the proof, to improve the clarity of the exposition, we denote the amplified comultiplication by $\Delta_0 : M_0 \to M_0^r \mathbin{\overline{\otimes}} M_0$. The relation between Δ_0 and Δ has been concretized in Remark 2.22.

Put $M^r := p(\mathrm{M}_n(\mathbb{C}) \otimes M)p$ so that literally $M_0^r \subset M^r$. Let $\mathcal{H} \subset L^2(M^r \mathbin{\overline{\otimes}} M) \ominus L^2(\Delta_0(M_0))$ be a $\Delta_0(M_0)$-$\Delta_0(M_0)$-subbimodule of finite left $\Delta_0(M_0)$-dimension. Using the notation of Remark 2.22, we put

$$\mathcal{H}' := Z\, (\mathrm{id} \otimes \mathrm{id} \otimes \zeta^{-1})(\mathcal{H} \otimes \mathrm{M}_n(\mathbb{C}))\, Z^*$$

and notice that $\mathcal{H}' \subset L^2(M^r \mathbin{\overline{\otimes}} M^r) \ominus L^2(\Delta(M_0^r))$ is a non-zero $\Delta(M_0^r)$-$\Delta(M_0^r)$-subbimodule of finite left $\Delta(M_0^r)$-dimension. To conclude the proof of the lemma, we have to find automorphisms $\beta_1, \ldots, \beta_k \in \mathrm{Aut}(M_0^r)$ and a unitary $\psi : \mathcal{H}' \to L^2(M_0^r)^{\oplus k} : \xi \mapsto (\psi_1(\xi), \ldots, \psi_k(x))$ such that

$$\psi_i(\Delta(x)\, \xi\, \Delta(y)) = x\, \psi_i(\xi)\, \beta_i(y), \text{ for all } x, y \in M_0^r, \xi \in \mathcal{H}', i = 1, \ldots, k,$$

and such that every β_i generates an infinite discrete subgroup of $\mathrm{Out}(M_0^r)$.

By our assumptions on $M_0 \subset M$, we can choose a subset $\mathcal{L}_0 \subset \mathcal{N}_M(M_0)$ such that

$$L^2(M) = L^2(M_0) \oplus \bigoplus_{V \in \mathcal{L}_0} L^2(M_0)V$$

and such that for every $V \in \mathcal{L}_0$, the automorphism $\mathrm{Ad}\, V$ of M_0 generates a discrete infinite subgroup of $\mathrm{Out}(M_0)$. Fix $V \in \mathcal{L}_0$. Take a partial isometry $v \in \mathrm{M}_n(\mathbb{C}) \otimes M_0$ such that $vv^* = p$ and $v^*v = (\mathrm{id} \otimes \mathrm{Ad}\, V)(p)$. Write $V' := v(1 \otimes V)$ and note that $V' \in \mathcal{N}_{M^r}(M_0^r)$. As such, we find a subset $\mathcal{L}_1 \subset \mathcal{N}_{M^r}(M_0^r)$ such that

$$L^2(M^r) = L^2(M_0^r) \oplus \bigoplus_{V \in \mathcal{L}_1} L^2(M_0^r)V$$

and such that for every $V \in \mathcal{L}_1$, the automorphism $\mathrm{Ad}\, V$ of M_0^r generates a discrete infinite subgroup of $\mathrm{Out}(M_0^r)$.

Define the subset $\mathcal{L}_2 \subset \mathcal{N}_{M^r \overline{\otimes} M^r}(M_0^r \overline{\otimes} M_0^r)$ given by

$$\mathcal{L}_2 := \{1 \otimes V \mid V \in \mathcal{L}_1\} \cup \{V \otimes 1 \mid V \in \mathcal{L}_1\} \cup \{V_1 \otimes V_2 \mid V_1, V_2 \in \mathcal{L}_1\} \; .$$

We get that

$$L^2(M^r \overline{\otimes} M^r) = L^2(M_0^r \overline{\otimes} M_0^r) \oplus \bigoplus_{W \in \mathcal{L}_2} L^2(M_0^r \overline{\otimes} M_0^r)W \; . \tag{6.3}$$

Also for every $W \in \mathcal{L}_2$, the automorphism $\mathrm{Ad}\, W$ of $M_0^r \overline{\otimes} M_0^r$ is of the form $\alpha_W \otimes \beta_W$, where at least one of the α_W, β_W generates a discrete infinite subgroup of $\mathrm{Out}(M_0^r)$.

Denote by P_0 the orthogonal projection of $L^2(M^r \overline{\otimes} M^r)$ onto the closed subspace $L^2(M_0^r \overline{\otimes} M_0^r)$ and define $\mathcal{H}_0$ as the closure of $P_0(\mathcal{H}')$. Then $\mathcal{H}_0 \subset L^2(M_0^r \overline{\otimes} M_0^r) \ominus L^2(\Delta(M_0^r))$ is a $\Delta(M_0^r)$-$\Delta(M_0^r)$-subbimodule of finite left dimension. By [IPV10, Proposition 7.2.3], we get that $\mathcal{H}_0 = \{0\}$.

For every $W \in \mathcal{L}_2$, denote by P_W the orthogonal projection of $L^2(M^r \overline{\otimes} M^r)$ onto the closed subspace $L^2(M_0^r \overline{\otimes} M_0^r)W$ and define

$$\varphi_W : L^2(M^r \overline{\otimes} M^r) \to L^2(M_0^r \overline{\otimes} M_0^r) : \varphi_W(\xi) = P_W(\xi)W^* \; .$$

Since W normalizes $M_0^r \overline{\otimes} M_0^r$ and since (6.3) is an orthogonal decomposition, we get that

$$\varphi_W(\Delta(x)\, \xi\, \Delta(y)) = \Delta(x)\, \varphi_W(\xi)\, (\alpha_W \otimes \beta_W)\Delta(y)$$

for all $x, y \in M_0^r$ and all $\xi \in L^2(M^r \mathbin{\overline{\otimes}} M^r)$. Denote by $\mathcal{H}_W$ the closure of $\varphi_W(\mathcal{H}')$. Below we prove the following statement: if $\mathcal{H}_W \neq \{0\}$, then there exists a unitary $\psi_W : \mathcal{H}_W \to L^2(M_0^r)$ and an automorphism $\gamma_W \in \mathrm{Aut}(M_0^r)$ such that

$$\psi_W(\Delta(x)\,\xi\,(\alpha_W \otimes \beta_W)\Delta(y)) = x\,\psi_W(\xi)\,\gamma_W(y) \tag{6.4}$$

for all $x, y \in M_0^r$ and all $\xi \in \mathcal{H}_W$, and such that γ_W generates a discrete infinite subgroup of $\mathrm{Out}(M_0^r)$. For the moment, we assume that the statement is proven and deduce the lemma from it. Whenever $\mathcal{H}_W \neq \{0\}$, we denote by $\mathcal{K}_W$ the M_0^r-M_0^r-bimodule $L^2(M_0^r)$ with bimodule action $x \cdot \xi \cdot y = x\xi\gamma_W(y)$. Then $\psi_W \circ \varphi_W : \mathcal{H}' \to \mathcal{K}_W$ is a bimodular map with dense range. So, $\mathcal{K}_W$ is isomorphic with a subbimodule of $\mathcal{H}'$. Since $\mathcal{H}'$ has finite left dimension and since $\mathcal{H}_0 = \{0\}$, it follows that $\mathcal{H}'$ is isomorphic with the direct sum of finitely many $\mathcal{K}_W$'s. This proves the lemma.

So it remains to prove the statement above. Assume that $\mathcal{H}_W \neq \{0\}$. By construction, $\mathcal{H}_W$ is a $\Delta(M_0^r)$-$(\alpha_W \otimes \beta_W)\Delta(M_0^r)$-subbimodule of $L^2(M_0^r \mathbin{\overline{\otimes}} M_0^r)$ of finite left dimension. By Theorem 2.3, we have $(\alpha_W \otimes \beta_W)\Delta(M_0^r) \prec \Delta(M_0^r)$. By Lemma 6.2, there exist characters $\omega, \mu : \Lambda \to \mathbb{T}$ and an automorphism $\delta \in \mathrm{Aut}(\Lambda)$ such that, after unitarily conjugating α_W and β_W, we have that $\alpha_W(v_s) = \omega(s)\,v_{\delta(s)}$ and $\beta_W(v_s) = \mu(s)\,v_{\delta(s)}$ for all $s \in \Lambda$. Note that $(\alpha_W \otimes \beta_W)\Delta(v_s) = \Delta(\gamma_W(v_s))$, where the automorphism $\gamma_W \in \mathrm{Aut}(M_0^r)$ is defined by the formula $\gamma_W(v_s) = \omega(s)\mu(s)\,v_{\delta(s)}$.

So $(\alpha_W \otimes \beta_W)\Delta(M_0^r) = \Delta(M_0^r)$. We get in particular that $\mathcal{H}_W$ is a non-zero $\Delta(M_0^r)$-$\Delta(M_0^r)$-subbimodule of $L^2(M_0^r \mathbin{\overline{\otimes}} M_0^r)$ that has finite left dimension. It then follows from [IPV10, Proposition 7.2.3] that $\mathcal{H}_W \subset L^2(\Delta(M_0^r))$. Since M_0^r is a factor and $\mathcal{H}_W \neq \{0\}$, we get that $\mathcal{H}_W = L^2(\Delta(M_0^r))$. We can thus define $\psi : \mathcal{H}_W \to L^2(M_0^r)$ as being Δ^{-1}. By construction, (6.4) holds. It remains to prove that γ_W generates an infinite discrete subgroup of $\mathrm{Out}(M_0^r)$.

We know that at least one of the α_W, β_W generates an infinite discrete subgroup of $\mathrm{Out}(M_0^r)$. Assume that this is the case for α_W. View α_W as an element of $\mathrm{Out}(M_0^r)$ and view $\widehat{\Lambda}$ as a compact subgroup of $\mathrm{Out}(M_0^r)$. Since $\alpha_W(v_s) = \omega(s)\,v_{\delta(s)}$ for all $s \in \Lambda$, we have that α_W normalizes $\widehat{\Lambda}$. Since $\widehat{\Lambda}$ is compact and since α_W generates an infinite discrete subgroup, it follows that $\widehat{\Lambda}$ and α_W together generate a copy of $\widehat{\Lambda} \rtimes \mathbb{Z}$ as a closed subgroup of $\mathrm{Out}(M_0^r)$. Since $\gamma_W \in \alpha_W\widehat{\Lambda}$, it then follows that also γ_W generates an infinite discrete subgroup of $\mathrm{Out}(M_0^r)$. $\square$

For later use, we end this section with yet another elementary lemma.

Lemma 6.4. *Let Λ be a countable group and $\Delta : L\Lambda \to L\Lambda \mathbin{\overline{\otimes}} L\Lambda$ the comultiplication given by $\Delta(v_s) = v_s \otimes v_s$ for all $s \in \Lambda$. If $\alpha, \beta \in \mathrm{Aut}(L\Lambda)$ are* *automorphisms that satisfy* $(\alpha \otimes id) \circ \Delta = \Delta \circ \beta$*, then there exists a character* $\omega : \Lambda \to \mathbb{T}$ *such that* $\alpha = \beta = \alpha_\omega$*, where* $\alpha_\omega(v_s) = \omega(s)v_s$ *for all* $s \in \Lambda$*.*

Proof. Since $\Delta(\beta(v_s)) = \alpha(v_s) \otimes v_s$, we see that $\alpha(v_s) \otimes v_s \in \Delta(\mathrm{L}\Lambda)$. This implies that $\alpha(v_s)$ must be a multiple of v_s, for all $s \in \Lambda$. So we find a character $\omega : \Lambda \to \mathbb{T}$ such that $\alpha = \alpha_\omega$. But then also $\beta = \alpha_\omega$. $\square$

Chapter 7

Proof of the main results

In this chapter we prove our main Theorem 1.2 stated in the introduction. Before proceeding to the proof, we first describe the framework and then prove a more general result (see Theorem 7.1 below) which implies immediately Theorem 1.2.

Throughout this chapter, Γ will be a countable group satisfying one of the following assumptions:

1. Γ is non-amenable, i.c.c., weakly amenable, belongs to class $\mathcal{S}$ and admits a bound on the orders of its finite subgroups;
2. Γ is a non-degenerate amalgamated free product $\Gamma_1 *_\Sigma \Gamma_2$ or a non-degenerate HNN extension $\mathrm{HNN}(\Gamma_1, \Sigma, \theta)$ with Σ malnormal in Γ_1, respectively $\{\Sigma, \theta(\Sigma)\}$ malnormal in Γ_1;
3. Γ is i.c.c., weakly amenable, has positive first ℓ^2-Betti number and admits a bound on the orders of its finite subgroups.

Let H be a non-trivial abelian group and $H_0 < H$ be a subgroup such that H/H_0 is either trivial or torsion-free. Define $\mathcal{H} := H^{(\Gamma)}$ and consider the homomorphism

$$p_H : H^{(\Gamma)} \to H : p_H(x) = \sum_{g \in \Gamma} x_g \; .$$

Denote $\mathcal{H}_0 := p_H^{-1}(H_0)$, $G := \Gamma \times \Gamma$ and consider the left-right wreath product $\mathcal{G} := \mathcal{H} \rtimes G$, with its subgroup $\mathcal{G}_0 := \mathcal{H}_0 \rtimes G$. Put $M_0 := \mathrm{L}\mathcal{G}_0$ and $M := \mathrm{L}\mathcal{G}$.

Theorem 7.1. *Let Λ be any countable group such that $\pi : L\Lambda \to (L\mathcal{G}_0)^r$ is a $*$-isomorphism, for some $r > 0$. Then $r = 1$ and $\Lambda \cong p_{H'}^{-1}(H_0') \rtimes G$ for* *some abelian group H', with subgroup $H'_0 < H'$, such that $|H_0| = |H'_0|$ and $H/H_0 \cong H'/H'_0$.*

More precisely, there exist group isomorphisms $\delta : \Lambda \to p_{H'}^{-1}(H'_0) \rtimes G$ and $\gamma : H'/H'_0 \to H/H_0$, a probability measure preserving isomorphism $\theta : \widehat{H'} \to \widehat{H}$ satisfying $\theta(k+\eta) = \widehat{\gamma}(k)+\theta(\eta)$, for all $k \in \widehat{H'/H'_0}$ and a.e. $\eta \in \widehat{H'}$, a character $\omega : \mathcal{G}_0 \to \mathbb{T}$ and a unitary $w \in L\mathcal{G}_0$ such that

$$\pi = \operatorname{Ad} w \circ \alpha_\omega \circ \pi_\theta \circ \pi_\delta,$$

where

- *$\pi_\delta : L\Lambda \to L\big(p_{H'}^{-1}(H'_0) \rtimes G\big)$ is the $*$-isomorphism given by $\pi_\delta(v_s) = u_{\delta(s)}$ for all $s \in \Lambda$;*
- *$\pi_\theta : L\big(p_{H'}^{-1}(H'_0) \rtimes G\big) \to L\big(p_H^{-1}(H_0) \rtimes G\big)$ is the natural $*$-isomorphism associated with an infinite tensor product of copies of θ ;*
- *α_ω is the automorphism of $L\mathcal{G}_0$ given by $\alpha_\omega(u_g) = \omega(g)u_g$ for all $g \in \mathcal{G}_0$.*

This whole chapter is devoted to the proof of Theorem 7.1, following closely the strategy of [IPV10] and using many results of [IPV10]. At the end, we will deduce Theorem 1.2, with case 1.2.1 corresponding to the special case where $H_0 = H$, and case 1.2.2 corresponding to $H_0 = \{e\}$.

We fix a countable group Λ, a positive number $r > 0$ and a $*$-isomorphism $\pi : \mathrm{L}\Lambda \to M_0^r$. To simplify notations, we do not explicitly write π and identify $M_0^r = \mathrm{L}\Lambda$.

Consider the amplified comultiplication $\Delta : M_0 \to (M_0 \overline{\otimes} M_0)^r$ defined in section 2.7. Note that the amplified homomorphism Δ is only defined up to unitary conjugacy (see Remark 2.22 for details). We concretely realize the amplification $(M \overline{\otimes} M)^r$ as $M^r \overline{\otimes} M$.

Our initial assumptions on the group Γ guarantee that it is not inner amenable: case 1 follows from [OP08, Proposotion 2.1] and [CS11, Proposition 1.7.5], case 2 follows from [Io12b, Corollary 6.2] and [DI12, Lemma 8.2] and case 3 follows from [CSU13, Corollary D]. By Proposition 2.23.(c), $\mathcal{G}_0$ and $\mathcal{G}$ are i.c.c. groups, and M_0, M are II_1 factors with $M_0' \cap M = \mathbb{C}1$. From Proposition 2.23.(a), we get that $M_0' \cap M^\omega = \mathbb{C}1$ and that M_0 does not have property Gamma, so that $\mathrm{Out}(M_0)$ is a Polish group.

We write $A := \mathrm{L}\mathcal{H}$ so that $M = A \rtimes G$. We also write $A_0 := \mathrm{L}\mathcal{H}_0$ so that $M_0 = A_0 \rtimes G$.

Recall that a countable subgroup $\mathcal{A}$ of a Polish group $\mathcal{B}$ is said to be *discrete* if there exists a neighborhood $\mathcal{U}$ of the identity e in $\mathcal{B}$ such that $\mathcal{U} \cap \mathcal{A} = \{e\}$.

We start by two general lemmas on the structure of M_0 and M. The first one is an immediate consequence of Popa's cocycle superrigidity theorem [Po06b, Theorem 1.1].

Lemma 7.2. *Let $\beta \in \operatorname{Aut}(M_0)$ and assume that there exists a non-zero vector $\xi_0 \in L^2(M_0)$ such that $\xi_0\beta(a) = a\xi_0$ for all $a \in A_0$. Then there exist a character $\omega : G \to \mathbb{T}$ and a unitary $v \in \mathcal{N}_M(M_0)$ such that $\beta = (\operatorname{Ad} v) \circ \alpha_\omega$, where the automorphism α_ω is defined as $\alpha_\omega(au_g) = \omega(g)\, au_g$ for all $a \in A_0$, $g \in G$.*

If moreover β generates a discrete infinite subgroup of $\operatorname{Out}(M_0)$, we have that $E_{M_0}(v) = 0$.

Proof. Taking the polar decomposition of ξ_0, we find a non-zero partial isometry $v_0 \in M_0$ such that $v_0\beta(a) = av_0$ for all $a \in A_0$. By Proposition 2.23.(c), we have that $A_0' \cap M_0 = A_0$ and hence, $v_0v_0^* \in A_0$ and $v_0^*v_0 \in \beta(A_0)$. Since $G \curvearrowright A_0$ is ergodic, we can extend v_0 to a unitary $v_1 \in \mathcal{U}(M_0)$ such that $v_1\beta(a)v_1^* = a$ for all $a \in A_0$. Put $\beta_1 = (\operatorname{Ad} v_1) \circ \beta$. Since $\beta_1(a) = a$ for all $a \in A_0$, we have $\beta_1(u_g) = \mu_g\, u_g$ for all $g \in G$, where $\mu_g \in \mathcal{U}(A_0)$ and $(\mu_g)_{g\in G}$ defines a 1-cocycle for the action $G \curvearrowright A_0$.

By Popa's cocycle superrigidity theorem [Po06b, Theorem 1.1] for the action $G \overset{\sigma}{\curvearrowright} A$, we find a unitary $v_2 \in \mathcal{U}(A)$ and a character $\omega : G \to \mathbb{T}$ such that $\mu_g = \omega(g)\, v_2^*\sigma_g(v_2)$ for all $g \in G$. It follows that $v_2\beta_1(x)v_2^* = \alpha_\omega(x)$ for all $x \in M_0$. In particular, $v_2 \in \mathcal{N}_M(M_0)$. Putting $v := v_1^*v_2^*$, we have that $v \in \mathcal{N}_M(M_0)$ and $\beta = (\operatorname{Ad} v) \circ \alpha_\omega$.

Finally, if β generates a discrete infinite subgroup of $\operatorname{Out}(M_0)$, we know that as an element of $\operatorname{Out}(M_0)$, β does not belong to the compact subgroup $\widehat{G} \subset \operatorname{Out}(M_0)$. So, $v \notin \mathcal{U}(M_0)$. Since $v \in \mathcal{N}_M(M_0)$ and $M_0' \cap M = \mathbb{C}1$, it follows that $E_{M_0}(v) = 0$. $\square$

Lemma 7.3. *Denote by H_e the copy of H inside $\mathcal{H}$ in position $e \in \Gamma$. Then M is generated by M_0 and the group of unitaries $\mathcal{L} := \{u_s \mid s \in H_e\}$ that normalize M_0. The image of $\mathcal{L}$ in $\operatorname{Out}(M_0)$ is a discrete subgroup of $\operatorname{Out}(M_0)$ that is isomorphic with H/H_0.*

Proof. By Proposition 2.23.(a), we know that $M_0' \cap M^\omega = \mathbb{C}1$. So whenever (a_n) is a sequence of unitaries in $\mathcal{U}(M)$ satisfying $\|xa_n - a_nx\|_2 \to 0$ for all $x \in M_0$, there exists a sequence $\lambda_n \in \mathbb{T}$ such that $\|a_n - \lambda_n 1\|_2 \to 0$. Assume that we have a sequence $s_n \in H_e$ such that $\operatorname{Ad}(u_{s_n})$, viewed as a sequence in $\operatorname{Out}(M_0)$, converges to the identity. We must prove that s_n belongs to $(H_0)_e$ eventually.

Since $\mathrm{Ad}(u_{s_n})$ converges to the identity in $\mathrm{Out}(M_0)$, we find a sequence of unitaries $w_n \in \mathcal{U}(M_0)$ such that $\mathrm{Ad}(w_n u_{s_n}) \to \mathrm{id}$ in $\mathrm{Aut}(M_0)$. This means that $\|x w_n u_{s_n} - w_n u_{s_n} x\|_2 \to 0$ for all $x \in M_0$. It follows that we can take a sequence $\lambda_n \in \mathbb{T}$ such that $\|w_n u_{s_n} - \lambda_n 1\|_2 \to 0$. So $\|u_{s_n} - \lambda_n w_n^*\|_2 \to 0$. In particular, we get that $\|u_{s_n} - E_{M_0}(u_{s_n})\|_2 \to 0$. Since $\|u_{s_n} - E_{M_0}(u_{s_n})\|_2 = 1$ whenever $s_n \notin (H_0)_e$, we conclude that $s_n \in (H_0)_e$ eventually. □

We now start a systematic study of the properties of the amplified comultiplication $\Delta : M_0 \to (M_0 \mathbin{\overline{\otimes}} M_0)^r$.

Lemma 7.4. *We have that* $\Delta(M_0)' \cap (M \mathbin{\overline{\otimes}} M)^r = \mathbb{C}1$.

Proof. This is an immediate consequence of Lemma 7.3, the assumption that H/H_0 is torsion-free and part (b) of Lemma 6.3. □

In what follows, we apply twice Theorem 5.4. So we need to check that the assumptions stated in the beginning of Section 5.2 are satisfied. Whenever Γ is as in assumption 3, we consider $(\beta_t)_{t\in\mathbb{R}}$ to be the Gaussian deformation defined in section 4.3.

Lemma 7.5. *Both when viewing M as the crossed product $M = B \rtimes (\{e\} \times \Gamma)$ with $B = A \rtimes (\Gamma \times \{e\})$, or as the crossed product $M = B \rtimes (\Gamma \times \{e\})$ with $B = A \rtimes (\{e\} \times \Gamma)$, all assumptions in the beginning of Section 5.2 are satisfied. More concretely, we have*

(a) $\Delta(M_0)' \cap (M \mathbin{\overline{\otimes}} M)^r = \mathbb{C}1$;

(b) If $\Gamma_0 < \Gamma$ is a subgroup of infinite index, then

$$M_0 \not\prec A \rtimes (\Gamma \times \Gamma_0) \quad \text{and} \quad M_0 \not\prec A \rtimes (\Gamma_0 \times \Gamma);$$

(c) M_0 is non-amenable relative to $A \rtimes (\Gamma \times \{e\})$ and non-amenable relative to $A \rtimes (\{e\} \times \Gamma)$, inside M. Moreover, in case 2, we also have that M_0 is non-amenable relative to $A \rtimes (\Gamma \times \Sigma)$ and non-amenable relative to $A \rtimes (\Sigma \times \Gamma)$;

(d) In case 3, we have that $\mathrm{id} \otimes \beta_t$ does not converge uniformly to id on the unit ball of $q\Delta(M_0)q$, for any non-zero projection $q \in \Delta(M_0)$.

Proof. Statement (a) is given by Lemma 7.4. Statement (b) is immediate. Statement (c) follows from Lemma 2.12 and Lemma 2.25 and statement (d) follows from Lemma 4.7. □

Lemma 7.6. *We have* $\Delta(A_0) \prec^f A \mathbin{\overline{\otimes}} A$.

Proof. Because of Lemma 7.5, we can apply Theorem 5.4 to the crossed product decompositions $M = (A \rtimes (\Gamma \times \{e\})) \rtimes \Gamma$ and $M = (A \rtimes (\{e\} \times \Gamma)) \rtimes \Gamma$, and the abelian (hence amenable) von Neumann subalgebra $\Delta(A_0) \subset M^r \mathbin{\overline{\otimes}} M$. We conclude that

$$\Delta(A_0) \prec^f M^r \mathbin{\overline{\otimes}} \big(A \rtimes (\Gamma \times \{e\})\big) \quad \textit{and} \quad \Delta(A_0) \prec^f M^r \mathbin{\overline{\otimes}} \big(A \rtimes (\{e\} \times \Gamma)\big)\ .$$

So by Lemma 2.7.(b), we get that $\Delta(A_0) \prec^f M^r \mathbin{\overline{\otimes}} A$. By symmetry, we also have that $\Delta(A_0) \prec^f A \mathbin{\overline{\otimes}} M^r$. Again by Lemma 2.7.(b), we conclude that $\Delta(A_0) \prec^f A \mathbin{\overline{\otimes}} A$. ☐

Lemma 7.7. *Let* $G_1 < G$ *be a subgroup of infinite index. Then we have that* $\Delta(LG) \not\prec M^r \mathbin{\overline{\otimes}} (A \rtimes G_1)$ *and* $\Delta(LG) \not\prec (A \rtimes G_1) \mathbin{\overline{\otimes}} M^r$.

Proof. By symmetry, it suffices to prove that $\Delta(LG) \not\prec M^r \mathbin{\overline{\otimes}} (A \rtimes G_1)$. Assume the contrary. A combination of Lemma 7.6 and Lemma 2.8 then gives that $\Delta(M_0) \prec M^r \mathbin{\overline{\otimes}} (A \rtimes G_1)$. Proposition 2.21.(c) now implies that $M_0 \prec A \rtimes G_1$, contradicting the assumption that $G_1 < G$ has infinite index. ☐

We can view M as the generalized Bernoulli crossed product $M = (\mathrm{L}H)^\Gamma \rtimes G$. As in Section 3.1, we may define the tensor length deformation by automorphisms $(\alpha_t)_{t\in\mathbb{R}}$ of the tracial von Neumann algebra $\widetilde{M} := (\mathrm{L}H * \mathrm{L}\mathbb{Z})^\Gamma \rtimes G$. Denote by $\delta : \Gamma \to \Gamma \times \Gamma$ the diagonal embedding.

Lemma 7.8. *Let* Γ *be as in assumption 1 or 3. Let* $Q \subset M^r \mathbin{\overline{\otimes}} M$ *be a von Neumann subalgebra such that for all non-zero projections* $p \in Q' \cap M^r \mathbin{\overline{\otimes}} M$*, we have that* Qp *is non-amenable relative to* $M^r \otimes 1$.

Assume that $\Delta(LG) \subset \mathcal{N}_{M^r \overline{\otimes} M}(Q)''$*. Then we have*

$$\sup_{b \in \mathcal{U}(Q' \cap M^r \overline{\otimes} M)} \|(id \otimes \alpha_t)(b) - b\|_2 \to 0 \quad \textit{as } t \to 0.$$

Proof. Since A is abelian and hence amenable, we have that Qp is non-amenable relative to $M^r \mathbin{\overline{\otimes}} A$, for all non-zero projections $p \in Q' \cap M^r \mathbin{\overline{\otimes}} M$. Denote $P := \mathcal{N}_{M^r \overline{\otimes} M}(Q)''$.

Assume that Γ is a non-amenable group in class $\mathcal{S}$ with the property that all finite subgroups of Γ have order at most $\kappa - 1$, for some fixed $\kappa \in \mathbb{N}$. We claim that the stabilizer $\operatorname{Stab}\mathcal{F}$ (with respect to the action $G \curvearrowright \Gamma$) is amenable whenever $\mathcal{F} \subset \Gamma$ satisfies $|\mathcal{F}| \geq \kappa$. Indeed, the stabilizer of κ distinct elements $s_1, \ldots, s_\kappa \in \Gamma$ equals $(e, s_1^{-1})\delta(\Gamma_0)(e, s_1)$, where $\Gamma_0 < \Gamma$ is defined as the centralizer of the elements $s_i s_1^{-1}$. These κ distinct elements $s_i s_1^{-1}$ necessarily generate an infinite subgroup of Γ. Since Γ belongs to class $\mathcal{S}$, the centralizer of an infinite subgroup is amenable (see Section 2.5). So Γ_0 is amenable and the claim that the stabilizer of $s_1, \dots, s_\kappa$ is amenable follows. So the lemma is a direct consequence of Theorem 3.1, even without using the assumption that $\Delta(LG) \subset \mathcal{N}_{M^r \overline{\otimes} M}(Q)''$.

Let now Γ be weakly amenable, with positive first ℓ^2-Betti number. Assume that all finite subgroups of Γ have order at most $\kappa - 1$, for some fixed $\kappa \in \mathbb{N}$.

By Theorem 3.1, it suffices to prove that for any finite subset $\mathcal{F} \subset \Gamma$, with $|\mathcal{F}| \geq \kappa$, we have that Q is strongly non-amenable relative to $M^r \overline{\otimes} (A \rtimes \operatorname{Stab} \mathcal{F})$. To prove this, assume by contradiction that there exist a finite subset $\mathcal{F} \subset \Gamma$, with $|\mathcal{F}| \geq \kappa$, and a non-zero projection $q \in Q' \cap M^r \overline{\otimes} M$ such that Qq is amenable relative to $M^r \overline{\otimes} (A \rtimes \operatorname{Stab} \mathcal{F})$.

Since Γ has positive first ℓ^2-Betti number, it is non-amenable and admits an unbounded 1-cocycle c into the left regular representation. We know from the previous case that, if $\mathcal{F} \subset \Gamma$ is a finite subset with $|\mathcal{F}| \geq \kappa$, then $\operatorname{Stab}(\mathcal{F})$ can be conjugate into $\delta(K_0)$, where K_0 is the centralizer of κ distinct elements in Γ. Since these κ distinct elements necessarily generate an infinite subgroup $K < \Gamma$ that commutes with K_0, by Lemma 4.1.1 it follows that either K_0 is amenable or the cocycle c is bounded on K. If the cocycle c is bounded on K, then since the left regular representation of Γ is mixing, by Lemma 4.1.2, we get that c is bounded on K_0. Thus, for any finite subset $\mathcal{F} \subset \Gamma$ with $|\mathcal{F}| \geq \kappa$ we have that $\operatorname{Stab} \mathcal{F}$ is conjugate to $\delta(K_0)$ and that either K_0 is amenable or the cocycle c is bounded on K_0.

If K_0 is amenable, then $\operatorname{Stab} \mathcal{F}$ is also amenable and we have that Qq is amenable relative to $M^r \overline{\otimes} A$, which contradicts our initial assumption.

If the cocycle c is bounded on K_0, then, by Theorem 5.5, one of the following statements must be true:

- There exists a non-zero projection $q' \in Q' \cap M^r \overline{\otimes} M$ such that Qq' is amenable relative to $M^r \overline{\otimes} A$;
- There exists a non-zero projection $r \in Z(Pq)$ such that $\mathrm{id} \otimes \beta_t \to \mathrm{id}$ uniformly on the unit ball of Pr.

The first alternative clearly contradicts the initial assumption. If $\mathrm{id} \otimes \beta_t \to \mathrm{id}$ uniformly on the unit ball of Pr, then since $\Delta(\mathrm{L}G)q \subset \mathcal{N}_{M^r \overline{\otimes} M}(Qq)''$, it follows that $\mathrm{id} \otimes \beta_t \to \mathrm{id}$ uniformly on the unit ball of $\Delta(\mathrm{L}G)q$. By Lemma 7.6 we get that, in particular, $\mathrm{id} \otimes \beta_t \to \mathrm{id}$ uniformly on the unit ball of $\Delta(A_0)$. Thus

$\mathrm{id} \otimes \beta_t \to \mathrm{id}$ uniformly on the set $\{q\Delta(au_g)q \mid a \in \mathcal{U}(A_0), g \in G\}$. Since $\{au_g \mid a \in \mathcal{U}(A_0), g \in G\}$ generates M_0, by Lemma 4.3, there exists a non-zero projection $q_1 \in \Delta(M_0)' \cap M \mathbin{\overline{\otimes}} M$ such that $\mathrm{id} \otimes \beta_t \to \mathrm{id}$ uniformly on the unit ball of $\Delta(M_0)q_1$, but this contradicts Lemma 4.7.

□

Lemma 7.9. *Suppose that Γ is as in assumption 2. Let $Q \subset M^r \mathbin{\overline{\otimes}} M$ be a von Neumann subalgebra and denote by P the von Neumann algebra generated by its normalizer in $M^r \mathbin{\overline{\otimes}} M$. Assume that Q is strongly non-amenable relative to $M^r \otimes 1$ and that $\Delta(LG) \subset P$. Then either*

$$\sup_{b \in \mathcal{U}(Q' \cap M^r \overline{\otimes} M)} \|(id \otimes \alpha_t)(b) - b\|_2 \to 0, \quad as\ t \to 0,$$

or Pq is amenable relative to $(M^r \mathbin{\overline{\otimes}} A) \rtimes (\Gamma \times \Sigma)$ or to $(M^r \mathbin{\overline{\otimes}} A) \rtimes (\Sigma \times \Gamma)$, for some non-zero projection $q \in P' \cap M^r \mathbin{\overline{\otimes}} M$.

Proof. We assume that P is strongly non-amenable relative to $(M^r \mathbin{\overline{\otimes}} A) \rtimes (\Gamma \times \Sigma)$ and to $(M^r \mathbin{\overline{\otimes}} A) \rtimes (\Sigma \times \Gamma)$ and we prove that $\mathrm{id} \otimes \alpha_t$ converges to id uniformly on $\mathcal{U}(Q' \cap M^r \mathbin{\overline{\otimes}} M)$.

Suppose first that $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ is non-degenerate and that Σ is malnormal in Γ_1. Let $g \in \Gamma$ be a non-trivial element. By Theorem 2.26, we have that the centralizer $Z_\Gamma(g)$ of g in Γ is either infinite cyclic or can be conjugate in Γ_1 or Γ_2. Thus, if $\mathcal{F} \subset \Gamma$ is a finite subset and $|\mathcal{F}| \geq 2$, then the stabilizer of $\mathcal{F}$ under the left-right multiplication action is either cyclic (and hence amenable) or it is conjugate to a subgroup of $\delta(\Gamma_i)$, for some $i = 1$ or 2.

Then the lemma follows from Theorem 3.1 once we have proven that Q is strongly non-amenable relative to $M^r \mathbin{\overline{\otimes}} (A \rtimes \delta(\Gamma_i))$, for $i = 1, 2$.

Assume, by contradiction, that there exists a non-zero projection $q \in Q' \cap M^r \mathbin{\overline{\otimes}} M$ such that Qq is amenable relative to $M^r \mathbin{\overline{\otimes}} (A \rtimes \delta(\Gamma_i))$. Denote $\mathcal{A} := M^r \mathbin{\overline{\otimes}} A$. By assumption, $\Delta(\mathrm{L}G) \subset P$ and moreover, by Lemma 2.13, we may assume that $q \in Z(P)$. Writing $M^r \mathbin{\overline{\otimes}} M$ as an amalgamated free product $M^r \mathbin{\overline{\otimes}} M = (\mathcal{A} \rtimes (\Gamma \times \Gamma_1)) *_{\mathcal{A} \rtimes (\Gamma \times \Sigma)} (\mathcal{A} \rtimes (\Gamma \times \Gamma_2))$ and applying [Va13, Theorem A], at least one of the following assertions is true:

- $Qq \prec \mathcal{A} \rtimes (\Gamma \times \Sigma)$;
- $Pq \prec \mathcal{A} \rtimes (\Gamma \times \Gamma_i)$, for some $i = 1$ or 2;
- Pq is amenable relative to $\mathcal{A} \rtimes (\Gamma \times \Sigma)$.

If $Pq \prec \mathcal{A} \rtimes (\Gamma \times \Gamma_i)$, for some $i = 1$ or 2, then by Lemma 7.6 and Lemma 2.8 it follows that $M_0 \prec A \rtimes (\Gamma \times \Gamma_i)$, which is impossible since Γ_i has infinite index in Γ, for both $i = 1$ and 2. Notice that, by assumption, the last alternative cannot hold.

If $Qq \prec \mathcal{A} \rtimes (\Gamma \times \Sigma)$, then we have that $Qq \prec \mathcal{A} \rtimes (\Gamma \times \{e\})$. To prove this, assume that $Qq \nprec \mathcal{A} \rtimes (\Gamma \times \{e\})$. Since $\Sigma < \Gamma$ is relatively malnormal, there exists an infinite index subgroup $\Lambda < \Gamma$ such that $\left|\Sigma \cap g\Sigma g^{-1}\right| < \infty$, for all $g \in \Gamma \setminus \Lambda$. Then, by [Va10b, Lemma 6.3], it follows that $\Delta(\mathrm{L}G) \prec \mathcal{A} \rtimes (\Gamma \times \Lambda)$ and hence, by Lemma 7.6 and Lemma 2.8, we get that $M_0 \prec A \rtimes (\Gamma \times \Lambda)$, which is impossible since Λ has infinite index in Γ.

By symmetry, writing $M^r \mathbin{\overline{\otimes}} M = (\mathcal{A} \rtimes (\Gamma_1 \times \Gamma)) *_{\mathcal{A} \rtimes (\Sigma \times \Gamma)} (\mathcal{A} \rtimes (\Gamma_2 \times \Gamma))$ and using the same arguments as above, it follows that also $Qq \prec \mathcal{A} \rtimes (\{e\} \times \Gamma)$ and hence, by Lemma 2.7 , $Qq \prec \mathcal{A}$. Now this implies that there exists a non-zero projection $q' \in Q' \cap M^r \mathbin{\overline{\otimes}} M$ such that Qq' is amenable relative to $M^r \mathbin{\overline{\otimes}} A$, which contradicts our initial assumption.

Suppose now that $\Gamma = \mathrm{HNN}(\Gamma_1, \Sigma, \theta)$ is non-degenerate and $\{\Sigma, \theta(\Sigma)\}$ is malnormal in Γ_1. A similar argument can be done also for HNN extensions, using Theorem 2.27 : if $\mathcal{F} \subset \Gamma$ is a finite subset with $|\mathcal{F}| \geq 2$, then $\mathrm{Stab}(\mathcal{F})$ is either infinite cyclic (and hence amenable) or conjugated to a subgroup of $\delta(\Gamma_1)$. Then the conclusion follows in the same manner as for amalgamated free products, using [Va13, Theorem 4.1] instead of [Va13, Theorem A]. □

The next lemma is an immediate consequence of [Io06, Lemma 2.4]

Lemma 7.10. *Let $p \in M^r \mathbin{\overline{\otimes}} A$ be a non-zero projection and $N \subset p(M^r \mathbin{\overline{\otimes}} A)p$ be a von Neumann subalgebra. If there exist $\delta > 0$ and $t > 0$ such that $\tau(w^*(id \otimes \alpha_t)(w)) \geq \delta$, for all $w \in \mathcal{U}(N)$, then there exists a finite subset $\mathcal{F} \subset \Gamma$ such that*

$$N \prec M^r \mathbin{\overline{\otimes}} (LH)^{\mathcal{F}}.$$

Lemma 7.11. *Write $Q := \Delta(A_0)' \cap M^r \mathbin{\overline{\otimes}} M$. Then $Q \prec^f A \mathbin{\overline{\otimes}} A$.*

Proof. It suffices to prove that there exists a non-zero projection $p \in Q' \cap M^r \mathbin{\overline{\otimes}} M$ such that

$$Qp \text{ is amenable relative to } M^r \otimes 1. \tag{7.1}$$

Indeed, suppose there exists a non-zero projection $p \in Q' \cap M^r \mathbin{\overline{\otimes}} M$ such that Qp is amenable relative to $M^r \otimes 1$. The normalizer of Q contains $\Delta(M_0)$ and by Lemma 7.4, we know that

$$\Delta(M_0)' \cap M^r \mathbin{\overline{\otimes}} M = \mathbb{C}1 .$$

So by Lemma 2.13, we conclude that Q is amenable relative to $M^r \otimes 1$. Applying twice Theorem 5.4, which is possible thanks to Lemma 7.5, it follows that $Q \prec^f M^r \overline{\otimes} (A \rtimes (\Gamma \times \{e\}))$ and that $Q \prec^f M^r \overline{\otimes} (A \rtimes (\{e\} \times \Gamma))$. It then follows from Lemma 2.7.(b) that $Q \prec^f M^r \overline{\otimes} A$. By symmetry, we also have that $Q \prec^f A \overline{\otimes} M^r$. Again using Lemma 2.7.(b), we reach the desired conclusion that $Q \prec^f A \overline{\otimes} A$.

Thus, the only thing we need to prove is the statement (7.1). Assume not, i.e. Q is strongly non-amenable relative to $M^r \otimes 1$. Under this assumption, we claim that $\mathrm{id} \otimes \alpha_t \to \mathrm{id}$ uniformly on $\mathcal{U}(\Delta(A_0))$. Denote $P := \mathcal{N}_{M^r \overline{\otimes} M}(Q)''$.

If Γ is as in assumptions 1 or 3, then since $\Delta(\mathrm{L}G) \subset P$, the claim follows immediately from Lemma 7.8.

Suppose now that Γ is an amalgamated free product $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ or an HNN extension $\Gamma = \mathrm{HNN}(\Gamma_1, \Sigma, \theta)$, as in assumption 2. Since $\Delta(\mathrm{L}G) \subset P$, Lemma 7.9 implies that either $\mathrm{id} \otimes \alpha_t \to \mathrm{id}$ uniformly on $\mathcal{U}(Q' \cap M^r \overline{\otimes} M)$ or there exists a non-zero projection $q \in P' \cap M^r \overline{\otimes} M$ such that Pq is amenable relative to $(M^r \overline{\otimes} A) \rtimes (\Gamma \times \Sigma)$ or to $(M^r \overline{\otimes} A) \rtimes (\Sigma \times \Gamma)$.

If $\mathrm{id} \otimes \alpha_t \to \mathrm{id}$ uniformly on $\mathcal{U}(Q' \cap M \overline{\otimes} M)$, then obviously $\mathrm{id} \otimes \alpha_t \to \mathrm{id}$ uniformly on $\mathcal{U}(\Delta(A_0))$, since all unitaries in $\Delta(A_0)$ commute with Q. If Pq is amenable relative to $(M^r \overline{\otimes} A) \rtimes (\Gamma \times \Sigma)$ or to $(M^r \overline{\otimes} A) \rtimes (\Sigma \times \Gamma)$, for some projection $q \in P' \cap M^r \overline{\otimes} M$, then since $\Delta(M_0) \subset P$, it follows that $\Delta(M_0)q$ is amenable relative to $(M^r \overline{\otimes} A) \rtimes (\Gamma \times \Sigma)$ or to $(M^r \overline{\otimes} A) \rtimes (\Sigma \times \Gamma)$. But both cases imply that Σ is co-amenable in Γ, which is not possible, by Lemma 2.25.

By Lemma 7.6, we have that $\Delta(A_0) \prec M^r \overline{\otimes} A$, i.e. there are non-zero projections $q \in \Delta(A_0)$, $p \in M^r \overline{\otimes} A$, a non-zero partial isometry $v \in p(M^r \overline{\otimes} M)q$ and a $*$-homomorphism $\theta : \Delta(A_0)q \to p(M^r \overline{\otimes} A)p$ such that $bv = v\theta(b)$, for all $b \in \Delta(A_0)q$.

Denote $N := \theta(\Delta(A_0)q) \subset p(M^r \overline{\otimes} A)p$. Then $q' := v^*v \in N' \cap p(M^r \overline{\otimes} A)p$ and we may assume that p is the support projection of $E_{M^r \overline{\otimes} A}(q')$. Since $q' \in N' \cap p(M^r \overline{\otimes} A)p$ and since $\mathrm{id} \otimes \alpha_t \to \mathrm{id}$ uniformly on $\mathcal{U}(\Delta(A_0))$, it follows that

$$id \otimes \alpha_t \to id \text{ uniformly on } (N)_1 q',$$

where $(N)_1$ denotes the unit ball of N.

Since $\sup_{x \in (N)_1} \left\|(\mathrm{id} \otimes \alpha_t)(E_{M^r \overline{\otimes} A}(x)) - E_{M^r \overline{\otimes} A}((\mathrm{id} \otimes \alpha_t)(x))\right\|_2 \to 0$, as $t \to 0$, we get that $id \otimes \alpha_t \to id$ uniformly on $E_{M^r \overline{\otimes} A}((N)_1 q') = (N)_1 E_{M^r \overline{\otimes} A}(q')$. Since p is the support of $E_{M^r \overline{\otimes} A}(q')$, we finally get that

$$id \otimes \alpha_t \to id \text{ uniformly on the unit ball of } N.$$

By Lemma 7.10, there exists a finite subset $\mathcal{F}$ of Γ such that $N \prec_{M^r \overline{\otimes} A} M^r \overline{\otimes} (\mathrm{L}H)^{\mathcal{F}}$, i.e. there are non-zero projections $q_1 \in N$ and $p_1 \in M^r \overline{\otimes} (\mathrm{L}H)^{\mathcal{F}}$, a non-zero partial isometry $v_1 \in p_1(M^r \overline{\otimes} A)q_1$ and a $*$-homomorphism $\theta_1 : Nq_1 \to p_1(M^r \overline{\otimes} (\mathrm{L}H)^{\mathcal{F}})p_1$ such that $xv_1 = v_1\theta_1(x)$, for all $x \in Nq_1$.

Notice that vv_1 is non-zero. Indeed, if $vv_1 = 0$, then $E_{M^r \overline{\otimes} A}(v^*v)v_1 = E_{M^r \overline{\otimes} A}(v^*vv_1) = 0$ and since p is the support of $E_{M^r \overline{\otimes} A}(v^*v)$, we get that $v_1 = pv_1 = E_{M^r \overline{\otimes} A}(v^*v)v_1 = 0$, contradiction.

Therefore $vv_1 \in p_1(M^r \overline{\otimes} M)q$ is a non-zero partial isometry and $\theta_1 \circ \theta : \Delta(A_0)q \to p_1(M^r \overline{\otimes} (\mathrm{L}H)^{\mathcal{F}})p_1$ is a $*$-homomorphism satisfying $xvv_1 = v\theta(x)v_1 = vv_1\theta_1(\theta(x))$, for all $x \in \Delta(A_0)q$, i.e.

$$\Delta(A_0) \prec M^r \overline{\otimes} (\mathrm{L}H)^{\mathcal{F}}.$$

Since A_0 is diffuse, by Proposition 2.21.(b), we get that $\Delta(A_0) \nprec M^r \otimes 1$ and hence, by [Io06, Lemma 1.5], it follows that $\Delta(M_0) \prec M^r \overline{\otimes} (A \rtimes \operatorname{Stab} \mathcal{F})$, but this contradicts Proposition 2.21.(c), since $\operatorname{Stab} \mathcal{F}$ has infinite index in $\Gamma \times \Gamma$. $\square$

Lemma 7.12. *There exists a unitary $\Omega \in (M \overline{\otimes} M)^r$ such that*

$$\Omega\, \Delta(LG)\, \Omega^* \subset (LG \overline{\otimes} LG)^r\ .$$

Proof. Also $M \overline{\otimes} M$ can be viewed as a generalized Bernoulli crossed product $M \overline{\otimes} M = (LH)^I \rtimes (G \times G)$, associated with $G \times G$ acting on the disjoint union $I := \Gamma \sqcup \Gamma$ of two copies of Γ. The corresponding tensor length deformation precisely is $\alpha_t \otimes \alpha_t \in \operatorname{Aut}(\widetilde{M} \overline{\otimes} \widetilde{M})$.

Denote by $\delta : \Gamma \to \Gamma \times \Gamma$ the diagonal embedding. Observe that the stabilizer (in $G \times G$) of an element $i \in I$ is either of the form $G \times g\delta(\Gamma)g^{-1}$ or $g\delta(\Gamma)g^{-1} \times G$, with $g \in G$. Since G is an i.c.c. group, the lemma will follow by applying Theorem 3.3 to the generalized Bernoulli action $G \times G \curvearrowright (LH)^I$, provided that we prove the following two statements.

1. $\sup_{g \in G} \|(\alpha_t \otimes \alpha_t)\Delta(u_g) - \Delta(u_g)\|_2 \to 0$, as $t \to 0$.
2. $\Delta(LG) \nprec M^r \overline{\otimes} (A \rtimes \delta(\Gamma))$ and $\Delta(LG) \nprec (A \rtimes \delta(\Gamma)) \overline{\otimes} M^r$.

Proof of 1. By symmetry, it suffices to prove that

$$\sup_{g \in G} \|(\mathrm{id} \otimes \alpha_t)\Delta(u_g) - \Delta(u_g)\|_2 \to 0, \quad \text{as} \quad t \to 0\ .$$

Since every $g \in G$ is the product of an element in $\Gamma \times \{e\}$ and an element in $\{e\} \times \Gamma$, again by symmetry, it suffices to prove that

$$\sup_{g \in \{e\} \times \Gamma} \|(\mathrm{id} \otimes \alpha_t)\Delta(u_g) - \Delta(u_g)\|_2 \to 0, \quad \text{as} \quad t \to 0 \,. \tag{7.2}$$

Denote $Q := \Delta(L(\Gamma \times \{e\}))$. By Proposition 2.21.(e), we have that Qp is non-amenable relative to $M^r \otimes 1$ for all non-zero projections $p \in Q' \cap M^r \mathbin{\overline{\otimes}} M$. The unitaries $\Delta(u_g)$, $g \in \{e\} \times \Gamma$, all commute with Q and the normalizer P of Q contains $\Delta(LG)$. If Γ is as in assumptions 1 and 3, then the statement (7.2) follows directly from Lemma 7.8.

If Γ is an amalgamated free product $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ or an HNN extension $\Gamma = \mathrm{HNN}(\Gamma_1, \Sigma, \theta)$, as in assumption 2, then Lemma 7.9 implies that either $\mathrm{id} \otimes \alpha_t \to \mathrm{id}$ uniformly on $\mathcal{U}(Q' \cap M^r \mathbin{\overline{\otimes}} M)$ or there exists a non-zero projection $q \in P' \cap M^r \mathbin{\overline{\otimes}} M$ such that Pq is amenable relative to $(M^r \mathbin{\overline{\otimes}} A) \rtimes (\Gamma \times \Sigma)$ or to $(M^r \mathbin{\overline{\otimes}} A) \rtimes (\Sigma \times \Gamma)$.

If $\mathrm{id} \otimes \alpha_t \to \mathrm{id}$ uniformly on $\mathcal{U}(Q' \cap M^r \mathbin{\overline{\otimes}} M)$, then (7.2) is proven. To finish the proof, we show that the second alternative gives rise to a contradiction. Note that, since $\Delta(\mathrm{L}G) \subset P$, it implies that $\Delta(\mathrm{L}G)q$ is amenable relative to $(M^r \mathbin{\overline{\otimes}} A) \rtimes (\Gamma \times \Sigma)$ or to $(M^r \mathbin{\overline{\otimes}} A) \rtimes (\Sigma \times \Gamma)$.

By Lemma 7.11 we know that $N := \Delta(A_0)' \cap M^r \mathbin{\overline{\otimes}} M \prec A \mathbin{\overline{\otimes}} A$. Then Lemma 2.5 implies that there exist a projection $p \in A^r \mathbin{\overline{\otimes}} A$ and an element $v \in \mathrm{M}_{1,n}(\mathbb{C}) \otimes (M^r \mathbin{\overline{\otimes}} M)p$ such that $vv^* = 1$, $v^*v = 1 \otimes p$ and $v^*Nv = \mathrm{M}_n(\mathbb{C}) \otimes (A \mathbin{\overline{\otimes}} A)p$. Note that, since $\Delta(A_0)$ is abelian, we have that $\Delta(A_0) \subset Z(N)$ and hence $v^*\Delta(A_0)v \subset 1 \otimes (A \mathbin{\overline{\otimes}} A)p$. Denote $\mathcal{G} := \{\Delta(u_g) \mid g \in G\}$ and remark that $\mathcal{G}$ is a group of unitaries normalizing N. Since $\Delta(\mathrm{L}G)q$ is amenable relative to $(M^r \mathbin{\overline{\otimes}} A) \rtimes (\Gamma \times \Sigma)$ or to $(M^r \mathbin{\overline{\otimes}} A) \rtimes (\Sigma \times \Gamma)$ and since $\Delta(M_0)' \cap M^r \mathbin{\overline{\otimes}} M = \mathbb{C}1$, applying Lemma 2.15 for the group of unitaries $v^*\mathcal{G}v$ normalizing v^*Nv, it follows that $v^*\Delta(M_0)v$ is amenable relative to $\mathrm{M}_n(\mathbb{C}) \mathbin{\overline{\otimes}} M^r \mathbin{\overline{\otimes}} (A \rtimes (\Gamma \times \Sigma))$ or to $\mathrm{M}_n(\mathbb{C}) \mathbin{\overline{\otimes}} M^r \mathbin{\overline{\otimes}} (A \rtimes (\Sigma \times \Gamma))$. This further implies that $\Delta(M_0)$ is amenable relative to $M^r \mathbin{\overline{\otimes}} (A \rtimes (\Gamma \times \Sigma))$ or to $M^r \mathbin{\overline{\otimes}} (A \rtimes (\Sigma \times \Gamma))$, and finally, we get that both cases imply the co-amenability of Σ in Γ, which contradicts Lemma 2.25.

Proof of 2. Since $\delta(\Gamma)$ has infinite index in G, statement 2 follows immediately from Lemma 7.7. ☐

Lemma 7.13. *If $\mathcal{H} \subset \Delta(A_0)' \cap (M \mathbin{\overline{\otimes}} M)^r$ is a finite-dimensional, globally $(\mathrm{Ad}\, \Delta(u_g))_{g \in G}$-invariant subspace, then $\mathcal{H} \subset \mathbb{C}1$.*

Proof. Put $\mathcal{H}' := \{x - E_{\Delta(M_0)}(x) \mid x \in \mathcal{H}\}$. The main part of the proof consists in showing that $\mathcal{H}' = \{0\}$. Assume on the contrary that $\mathcal{H}' \neq \{0\}$. Note that $\mathcal{H}'$ is a finite-dimensional, globally $(\operatorname{Ad}\Delta(u_g))_{g\in G}$-invariant subspace of $\Delta(A_0)'\cap(M\mathbin{\overline{\otimes}}M)^r$ and that $\mathcal{H}'\subset(M\mathbin{\overline{\otimes}}M)^r\ominus\Delta(M_0)$.

Denote by $\mathcal{K}$ the closed linear span of $\Delta(M_0)\mathcal{H}'$ inside $L^2((M\mathbin{\overline{\otimes}}M)^r)$. Observe that $\mathcal{K}$ is a $\Delta(M_0)$-$\Delta(M_0)$-subbimodule of $L^2((M\mathbin{\overline{\otimes}}M)^r)\ominus L^2(\Delta(M_0))$ that has finite left dimension. By Lemma 7.3, the assumption that H/H_0 is torsion-free, and Lemma 6.3, there exist automorphisms $\beta_1,\ldots,\beta_k\in\operatorname{Aut}(M_0)$ and a unitary $\psi:\mathcal{K}\to L^2(M_0)^{\oplus k}:\xi\mapsto(\psi_1(\xi),\ldots,\psi_k(\xi))$ such that

$$\psi_i(\Delta(x)\,\xi\,\Delta(y))=x\,\psi_i(\xi)\,\beta_i(y)\quad\text{for all }\ x,y\in M_0,\xi\in\mathcal{K},i=1,\ldots,k,$$

and such that every β_i generates a discrete infinite subgroup of $\operatorname{Out}(M_0)$.

Fix an $i\in\{1,\ldots,k\}$ and note that $\psi_i(\mathcal{H}')\neq\{0\}$. Take a non-zero vector $\xi_0\in\psi_i(\mathcal{H}')$. Since the elements of $\mathcal{H}'$ commute with $\Delta(A_0)$, it follows that that $\xi_0\beta_i(a)=a\xi_0$ for all $a\in A_0$. By Lemma 7.2, we then find a unitary $v\in\mathcal{N}_M(M_0)$ and a character $\omega:G\to\mathbb{T}$ such that $\beta_i=(\operatorname{Ad}v)\circ\alpha_\omega$ and such that $E_{M_0}(v)=0$. Recall that $\alpha_\omega(au_g)=\omega(g)\,au_g$ for all $a\in A_0$ and all $g\in G$.

Put $\mathcal{H}_i':=\psi_i(\mathcal{H}')v$ and note that $\mathcal{H}_i'$ is a finite-dimensional subspace of $L^2(M)$ such that $\xi a=a\xi$ for all $\xi\in\mathcal{H}_i'$, $a\in A_0$ and such that $\mathcal{H}_i'$ is globally invariant under $\xi\mapsto\overline{\omega(g)}\,u_g\xi u_g^*$ for all $g\in G$. By Proposition 2.23.(c), we have $A_0'\cap M=A$. So $\mathcal{H}_i'\subset L^2(A)$. It follows that $\mathcal{H}_i'$ is a finite-dimensional subspace of $L^2(A)$ that is globally invariant under the generalized Bernoulli action $G\curvearrowright A$. By Lemma 2.20, the latter is weakly mixing. It follows that $\mathcal{H}_i'\subset\mathbb{C}1$. So, $\psi_i(\mathcal{H}')\subset\mathbb{C}v^*$. Since $\psi_i(\mathcal{H}')\subset L^2(M_0)$, while v^* is orthogonal to $L^2(M_0)$, we find that $\psi_i(\mathcal{H}')=\{0\}$, which is absurd.

So we have proven that $\mathcal{H}'=\{0\}$, meaning that $\mathcal{H}\subset\Delta(M_0)$. So $\mathcal{H}=\Delta(\mathcal{H}_0)$ where $\mathcal{H}_0\subset A_0'\cap M_0$ is a finite-dimensional, globally $(\operatorname{Ad}u_g)_{g\in G}$-invariant subspace. Since $A_0'\cap M_0=A_0$ and since the action $G\curvearrowright A_0$ is weakly mixing, it follows that $\mathcal{H}_0\subset\mathbb{C}1$. Then also $\mathcal{H}\subset\mathbb{C}1$. □

Lemma 7.14. *We have that $r=1$ and that there exist a unitary $v\in M_0$, a character $\omega:G\to\mathbb{T}$ and an injective group homomorphism $\rho:G\to\Lambda$ such that*

$$\omega(g)\,vu_gv^*=v_{\rho(g)}\ \textit{for all}\ g\in G\quad\textit{and}\quad\Delta(vA_0v^*)\subset vA_0v^*\mathbin{\overline{\otimes}}vA_0v^*\;.$$

Proof. We view $M\mathbin{\overline{\otimes}}M$ as the crossed product $M\mathbin{\overline{\otimes}}M=(A\mathbin{\overline{\otimes}}A)\rtimes(G\times G)$. By Proposition 2.23.(c), we have that $(A\mathbin{\overline{\otimes}}A)'\cap(M\mathbin{\overline{\otimes}}M)=A\mathbin{\overline{\otimes}}A$, meaning that the generalized Bernoulli action $G\times G\curvearrowright A\mathbin{\overline{\otimes}}A$ is essentially free. By Lemma 7.12 and after a unitary conjugacy of Δ, we have $\Delta(LG)\subset(LG\mathbin{\overline{\otimes}}LG)^r$. Put $C:=\Delta(A_0)'\cap(M\mathbin{\overline{\otimes}}M)^r$.

From Lemma 7.11, we know that $C \prec^f A \mathbin{\overline{\otimes}} A$. By construction, the unitaries $\Delta(u_g)$, with $g \in G$, normalize C. By Lemma 7.13, the action $(\operatorname{Ad} \Delta(u_g))_{g \in G}$ on the center $\mathcal{Z}(C)$ of C is weakly mixing. Actually, Lemma 7.13 says that even the action $(\operatorname{Ad} \Delta(u_g))_{g \in G}$ on C has no non-trivial finite-dimensional invariant subspaces. This means that all the assumptions of [IPV10, Theorem 6.1] are satisfied. Denote by N the von Neumann algebra generated by C and the unitaries $(\Delta(u_g))_{g \in G}$. Then $\Delta(M_0) \subset N$ and it follows from Proposition 2.21.(c) that $N \not\prec M \mathbin{\overline{\otimes}} (A \rtimes G_1)$ and $N \not\prec (A \rtimes G_1) \mathbin{\overline{\otimes}} M$ whenever $G_1 < G$ has infinite index. So also all the assumptions of [IPV10, Corollary 6.2] are satisfied. From [IPV10, Theorem 6.1 and Corollary 6.2], it then follows that $r = 1$ and that there exist a unitary $\Omega_1 \in M \mathbin{\overline{\otimes}} M$, a character $\omega : G \to \mathbb{T}$ and group homomorphisms $\gamma_1, \gamma_2 : G \to G$ such that

$$\Omega_1 \Delta(A_0) \Omega_1^* \subset A \mathbin{\overline{\otimes}} A \quad \text{and} \quad \Omega_1 \Delta(u_g) \Omega_1^* = \omega(g)\, u_{\gamma_1(g)} \otimes u_{\gamma_2(g)} \; . \tag{7.3}$$

Since $r = 1$, we may from now on assume that $M_0 = L\Lambda$ and that $\Delta : M_0 \to M_0 \mathbin{\overline{\otimes}} M_0$ is the original comultiplication given by $\Delta(v_s) = v_s \otimes v_s$ for all $s \in \Lambda$.

By (7.3) and Lemma 7.7, the ranges of γ_1 and γ_2 are finite index subgroups of G. Denote by $\zeta : M \mathbin{\overline{\otimes}} M \to M \mathbin{\overline{\otimes}} M : \zeta(x \otimes y) = y \otimes x$ the flip automorphism. Since $\zeta \circ \Delta = \Delta$, it follows from (7.3) that

$$(u_{\gamma_2(g)} \otimes u_{\gamma_1(g)})\; \zeta(\Omega_1)\, \Omega_1^* \; (u_{\gamma_1(g)}^* \otimes u_{\gamma_2(g)}^*) = \zeta(\Omega_1)\, \Omega_1^* \quad \text{for all } \; g \in G \; .$$

Because G is i.c.c. and because the subgroups $\gamma_1(G) < G$ and $\gamma_2(G) < G$ have finite index, we get that $\{(\gamma_2(g) x \gamma_2(g)^{-1}, \gamma_1(g) y \gamma_1(g)^{-1}) \mid g \in G\}$ is an infinite set for all $(x,y) \in (\mathcal{G} \times \mathcal{G}) - \{e\}$. By Lemma 6.1, we then find an $h \in G$ such that $\gamma_1(g) = h \gamma_2(g) h^{-1}$ for all $g \in G$. This means that after replacing Ω_1 by $(1 \otimes u_h)\Omega_1$, we may assume that $\gamma_1 = \gamma_2$. We denote this homomorphism as γ. It then also follows that $\zeta(\Omega_1)$ is a multiple of Ω_1. Since $\Delta(u_g)$ and $u_{\gamma(g)} \otimes u_{\gamma(g)}$ are unitarily conjugate, the homomorphism γ is injective.

Define $K_0 := \widehat{H/H_0}$ and identify K_0 with the group of characters on H that are equal to 1 on H_0. Whenever $\eta \in K_0$, the formula

$$\widetilde{\eta} : xg \mapsto \eta\Big(\sum_{h \in \Gamma} x_h\Big) \quad \text{for all } \; x \in H^{(\Gamma)}, g \in G \; ,$$

defines a character on $\mathcal{G}$ and hence an automorphism $\alpha_\eta \in \operatorname{Aut}(M)$ by the formula $\alpha_\eta(u_z) = \widetilde{\eta}(z) u_z$ for all $z \in \mathcal{G}$. Since η equals 1 on H_0, we get that $\alpha_\eta(a) = a$ for all $a \in M_0$. More precisely, $(\alpha_\eta)_{\eta \in K_0}$ is a continuous action of K_0 on M and the fixed point algebra of this action equals M_0.

Let $\eta, \eta' \in K_0$. Applying $\alpha_\eta \otimes \alpha_{\eta'}$ to (7.3), it follows that $\Omega_1^*(\alpha_\eta \otimes \alpha_{\eta'})(\Omega_1)$ commutes with $u_{\gamma(g)} \otimes u_{\gamma(g)}$ for all $g \in G$. Since G is i.c.c. and $\gamma(G) < G$ has finite index, we have that $\{(\gamma(g)x\gamma(g)^{-1}, \gamma(g)y\gamma(g)^{-1}) \mid g \in G\}$ is an infinite set for all $(x,y) \in (\mathcal{G}\times\mathcal{G})-\{e\}$. Using Lemma 2.20, it follows that $\Omega_1^*(\alpha_\eta \otimes \alpha_{\eta'})(\Omega_1)$ must be a multiple of 1 and we find $\Psi(\eta,\eta') \in \mathbb{T}$ such that

$$(\alpha_\eta \otimes \alpha_{\eta'})(\Omega_1) = \Psi(\eta,\eta')\,\Omega_1 \quad \text{for all } (\eta,\eta') \in K_0 \times K_0 \; . \tag{7.4}$$

It follows that Ψ is a continuous character on $K_0 \times K_0$. Since $\zeta(\Omega_1)$ is a multiple of Ω_1, we also get that $\Psi(\eta,\eta') = \Psi(\eta',\eta)$ for all $(\eta,\eta') \in K_0 \times K_0$. Since $\widehat{K_0} = H/H_0$, we find an $x \in H$ such that $\Psi(\eta,\eta') = \eta(x)\eta'(x)$ for all $(\eta,\eta') \in K_0 \times K_0$.

For every $g \in \Gamma$, denote by $\pi_g : \mathrm{L}H \to (\mathrm{L}H)^\Gamma$ the embedding of $\mathrm{L}H$ as the g-th tensor factor. Write $V_x := \pi_e(u_x)$ and put $\Omega_2 := (V_x^* \otimes V_x^*)\Omega_1$. From (7.4), it follows that $\Omega_2 \in M_0 \mathbin{\overline{\otimes}} M_0$. Denote by $x_e \in H^{(\Gamma)}$ the element $x \in H$ viewed in position e. Define the injective group homomorphism $\gamma' : G \to \mathcal{G}_0$ by $\gamma'(g) = x_e^{-1}\gamma(g)x_e$. It follows from (7.3) that

$$\Omega_2^*(u_{\gamma'(g)} \otimes u_{\gamma'(g)})\Omega_2 = \Delta(\overline{\omega(g)}u_g) \quad \text{for all } g \in G \; . \tag{7.5}$$

Since $\gamma(G)$ has finite index in G and since G is i.c.c., we have that $\{\gamma'(g)x\gamma'(g)^{-1} \mid g \in G\}$ is an infinite set for all $x \in \mathcal{G}_0 - \{e\}$. By Lemma 2.20, we get that the representation $(\mathrm{Ad}(u_{\gamma'(g)}))_{g\in G}$ on $L^2(M_0)\ominus\mathbb{C}1$ is weakly mixing. It then follows from (7.5) and [IPV10, Lemma 3.4] that there exist unitaries $w, v \in M_0$, a character $\omega' : G \to \mathbb{T}$ and an injective group homomorphism $\rho : G \to \Lambda$ such that

$$w u_{\gamma'(g)} w^* = \omega'(g)\, v_{\rho(g)}, \text{ for all } g \in G, \quad \text{and} \quad \Omega_2 = (w^* \otimes w^*)\Delta(v) \; .$$

In combination with (7.5), we get that

$$\begin{aligned}
\omega'(g)^2\,\Delta(v_{\rho(g)}) \quad &= \omega'(g)\, v_{\rho(g)} \;\otimes\; \omega'(g)\, v_{\rho(g)} \\
&= w u_{\gamma'(g)} w^* \otimes w u_{\gamma'(g)} w^* \\
&= \Delta(\overline{\omega(g)}\, v u_g v^*),
\end{aligned}$$

for all $g \in G$. So also $v u_g v^* = \omega(g)\omega'(g)^2\, v_{\rho(g)}$. This implies that

$$u_{\gamma'(g)}^*\, w^* v\, u_g = \omega(g)\omega'(g)\, w^* v \quad \text{for all } g \in G \; .$$

Lemma 6.1 then provides an element $k \in \mathcal{G}_0$ such that $\gamma'(g) = kgk^{-1}$ for all $g \in G$. It follows that $u_k^* w^* v \in \mathbb{C}1$ and that $\omega' = \overline{\omega}$. So, w is a multiple of $v u_k^*$ and

$$\omega(g)\, v u_g v^* = v_{\rho(g)} \quad \text{for all } g \in G \; .$$

From (7.3), we know that $\Omega_2\Delta(A_0)\Omega_2^* \subset A_0 \mathbin{\overline{\otimes}} A_0$. Since $\Omega_2 = (w^* \otimes w^*)\Delta(v)$ and since w is a multiple of vu_k^*, we conclude that

$$\Delta(vA_0v^*) \subset vA_0v^* \mathbin{\overline{\otimes}} vA_0v^* .$$

□

Proof of Theorem 7.1. The proof consists of three different parts.

Writing Λ as a semidirect product $\Sigma \rtimes G$

We do not explicitly write the isomorphism $\pi : \mathrm{L}\Lambda \to (\mathrm{L}\mathcal{G}_0)^r$, but directly identify $\mathrm{L}\Lambda = \mathrm{L}(\mathcal{G}_0)^r$. We denote by $\Delta : \mathrm{L}\Lambda \to \mathrm{L}\Lambda \mathbin{\overline{\otimes}} \mathrm{L}\Lambda$ the comultiplication given by $\Delta(v_s) = v_s \otimes v_s$ for all $s \in \Lambda$. Recall from [IPV10, Lemma 7.1] that a von Neumann subalgebra $P \subset \mathrm{L}\Lambda$ satisfies $\Delta(P) \subset P \mathbin{\overline{\otimes}} P$ if and only if $P = \mathrm{L}S$ for a subgroup $S < \Lambda$.

As above, we denote $\mathcal{H} := H^{(\Gamma)}$ and $\mathcal{H}_0 := p_H^{-1}(H_0)$. We write $G := \Gamma \times \Gamma$ and $A := \mathrm{L}\mathcal{H} = (\mathrm{L}H)^\Gamma$, with its subalgebra $A_0 := \mathrm{L}\mathcal{H}_0$. Finally, we put $M := A \rtimes G = \mathrm{L}(\mathcal{H} \rtimes G)$ and $M_0 := A_0 \rtimes G = \mathrm{L}\mathcal{G}_0$. For every character $\omega : \mathcal{G}_0 \to \mathbb{T}$, we denote by α_ω the induced automorphism of M_0 given by $\alpha_\omega(u_x) = \omega(x)u_x$ for all $x \in \mathcal{G}_0$.

By Lemma 7.14, we get that $r = 1$ and that we can compose the identification $\mathrm{L}\Lambda = \mathrm{L}\mathcal{G}_0$ with an inner automorphism of $\mathrm{L}\mathcal{G}_0$ and an automorphism of the form α_ω for a character $\omega : G \to \mathbb{T}$ such that after these compositions, we have

$$\Delta(A_0) \subset A_0 \mathbin{\overline{\otimes}} A_0 \quad \text{and} \quad u_g = v_{\rho(g)} \;\text{ for all }\; g \in G \;, \tag{7.6}$$

where $\rho : G \to \Lambda$ is an injective group homomorphism. It follows that $A_0 = \mathrm{L}\Sigma$ for an abelian subgroup $\Sigma < \Lambda$ and that we have written Λ as a semidirect product $\Lambda = \Sigma \rtimes G$, where G acts on Σ by group automorphisms. So from now on, we may assume that $\Lambda = \Sigma \rtimes G$ in such a way that $\mathrm{L}\Sigma = A_0$ and $v_g = u_g$ for all $g \in G$ (denoting as above by $(v_s)_{s\in\Lambda}$ the canonical unitaries for $\mathrm{L}\Lambda$, and by $(u_g)_{g\in\mathcal{G}_0}$ the canonical unitaries for $\mathrm{L}\mathcal{G}_0$).

Proving that Σ is of the form $p_{H'}^{-1}(H_0')$

Whenever we view Γ as the index set of the infinite tensor product $A = (\mathrm{L}H)^\Gamma$, we denote the elements of Γ by the letters i, j. We denote by $g \cdot i$ the left-right action of $g \in G$ on $i \in \Gamma$. We denote by $\pi_i : \mathrm{L}H \to (\mathrm{L}H)^\Gamma$ the embedding of $\mathrm{L}H$ into $(\mathrm{L}H)^\Gamma$ as the i-th tensor factor. We denote by $(\sigma_g)_{g\in G}$ the generalized

Bernoulli action given by $\sigma_g \circ \pi_i = \pi_{g\cdot i}$. We finally denote by $\delta : \Gamma \to G$, $\delta(g) = (g, g)$ the diagonal embedding. Since Γ is i.c.c., we have that $\delta(\Gamma) \cdot i$ is infinite for all $i \in \Gamma - \{e\}$. By Lemma 2.20, the action $(\sigma_{\delta(g)})_{g\in\Gamma}$ on $(\mathrm{L}H)^{\Gamma-\{e\}}$ is weakly mixing and we have that

$$\pi_e(\mathrm{L}H_0) = \{a \in A_0 \mid \sigma_{\delta(g)}(a) = a,\ g \in \Gamma\}\ , \tag{7.7}$$

$$\pi_e(\mathrm{L}H_0) \mathbin{\overline{\otimes}} \pi_e(\mathrm{L}H_0) = \{a \in A_0 \mathbin{\overline{\otimes}} A_0 \mid (\sigma_{\delta(g)} \otimes \sigma_{\delta(g)})(a) = a,\ g \in \Gamma\}\ , \tag{7.8}$$

$$\pi_e(\mathrm{L}H) \mathbin{\overline{\otimes}} \pi_e(\mathrm{L}H) = \{a \in A \mathbin{\overline{\otimes}} A \mid (\sigma_{\delta(g)} \otimes \sigma_{\delta(g)})(a) = a,\ g \in \Gamma\}\ . \tag{7.9}$$

For the rest of the proof, we only consider the comultiplication Δ restricted to $\mathrm{L}\Sigma$. Since $v_g = u_g$ for all $g \in G$, we have that $\Delta \circ \sigma_g = (\sigma_g \otimes \sigma_g) \circ \Delta$ for all $g \in G$. Using (7.7) and (7.8), it then follows that $\Delta(\pi_e(\mathrm{L}H_0)) \subset \pi_e(\mathrm{L}H_0) \mathbin{\overline{\otimes}} \pi_e(\mathrm{L}H_0)$. This means that we find an abelian group $H_1' < \Sigma$ with corresponding comultiplication $\Delta_1 : \mathrm{L}H_1' \to \mathrm{L}H_1' \mathbin{\overline{\otimes}} \mathrm{L}H_1'$, and an identification $\mathrm{L}H_1' = \mathrm{L}H_0$ such that $\Delta \circ \pi_e = (\pi_e \otimes \pi_e) \circ \Delta_1$. Composing with $(\sigma_g \otimes \sigma_g)_{g\in G}$, it follows that $\Delta \circ \pi_i = (\pi_i \otimes \pi_i) \circ \Delta_1$ for all $i \in \Gamma$. So we can view π_i as well as an injective group homomorphism of H_1' into Σ. Since the von Neumann algebras $\pi_i(\mathrm{L}H_0)$, $i \in \Gamma$, are in tensor product position inside $\mathrm{L}\Sigma$, it follows that the subgroups $\pi_i(H_1') < \Sigma$, $i \in \Gamma$, are in direct sum position inside Σ.

Fix an element $x \in H$. The formula $\Omega_x(g) := \pi_e(u_x)\pi_{g\cdot e}(u_x^*)$ defines a 1-cocycle for the action $(\sigma_g)_{g\in G}$ on A_0. Hence $g \mapsto \Delta(\Omega_x(g))$ is a 1-cocycle for the generalized Bernoulli action $(\sigma_g \otimes \sigma_g)_{g\in G}$ on $(\mathrm{L}H)^\Gamma \mathbin{\overline{\otimes}} (\mathrm{L}H)^\Gamma$. By Popa's cocycle superrigidity theorem [Po06b, Theorem 1.1], we find a unitary $\mathcal{V}_x \in (\mathrm{L}H)^\Gamma \mathbin{\overline{\otimes}} (\mathrm{L}H)^\Gamma$ such that

$$\Delta(\Omega_x(g)) = \mathcal{V}_x\, (\sigma_g \otimes \sigma_g)(\mathcal{V}_x^*) \quad \text{for all}\ \ g \in G\ .$$

By construction, $\Omega_x(\delta(g)) = 1$ for all $g \in \Gamma$. From (7.9), it then follows that $\mathcal{V}_x = (\pi_e \otimes \pi_e)(\mathcal{U}_x)$ for a unitary $\mathcal{U}_x \in \mathrm{L}H \mathbin{\overline{\otimes}} \mathrm{L}H$. So we get that

$$\Delta(\pi_e(u_x)\pi_{g\cdot e}(u_x^*)) = (\pi_e \otimes \pi_e)(\mathcal{U}_x)\, (\pi_{g\cdot e} \otimes \pi_{g\cdot e})(\mathcal{U}_x^*) \quad \text{for all}\ \ x \in H, g \in G\ .$$

Applying $\sigma_h \otimes \sigma_h$ for an arbitrary $h \in G$, and combining with the earlier definition of Δ_1, we find that for all $x \in H$ and $i, j \in \Gamma$

$$\begin{aligned} &\Delta((\pi_i \otimes \pi_j)(u_x \otimes u_x^*)) = (\pi_i \otimes \pi_i)(\mathcal{U}_x)\, (\pi_j \otimes \pi_j)(\mathcal{U}_x^*) \\ &\Delta \circ \pi_i = (\pi_i \otimes \pi_i) \circ \Delta_1. \end{aligned} \tag{7.10}$$

Define $H_2 := \{(x, y) \in H \times H \mid x + y \in H_0\}$. Then H_2 is generated by the subgroups $H_0 \times H_0$ and $\{(x, -x) \mid x \in H\}$. Since $\mathrm{L}H_1' = \mathrm{L}H_0$, the von Neumann algebra generated by the elements $\{(\pi_i \otimes \pi_j)(u_x \otimes u_x^*) \mid i, j \in \Gamma, x \in H\}$, together with the algebras $\pi_i(\mathrm{L}H_1')$, $i \in \Gamma$, equals the von Neumann algebra generated by all the $(\pi_i \otimes \pi_j)(\mathrm{L}H_2)$, which is the whole of $A_0 = \mathrm{L}\Sigma$.

So the formulae in (7.10) entirely determine Δ. Also note that for a given $x \in H$, the unitary $\mathcal{U}_x$ is uniquely determined up to multiplication by a scalar in $\mathbb{T}$. Finally observe that for $x \in H_0$, we have $\mathcal{U}_x = \Delta_1(u_x)$, up to multiplication by a scalar in $\mathbb{T}$. In particular, $\mathcal{U}_x \in \mathrm{L}H_0 \mathbin{\overline{\otimes}} \mathrm{L}H_0$ whenever $x \in H_0$.

For all distinct $i, j \in \Gamma$, denote by $\pi_{ij} : \mathrm{L}H_2 \to A_0$ the embedding into the i'th and j'th coordinate. It follows from (7.10) that we can identify $\mathrm{L}H_2 = \mathrm{L}H_2'$ for some abelian group H_2' with the corresponding comultiplication $\Delta_2 : \mathrm{L}H_2' \to \mathrm{L}H_2' \mathbin{\overline{\otimes}} \mathrm{L}H_2'$ given by the following formulae that use the tensor leg numbering notation:

$$
\begin{aligned}
&\Delta_2(u_x \otimes u_x^*) = (\mathcal{U}_x)_{13}\,(\mathcal{U}_x^*)_{24} \quad \text{for all } \ x \in H \ , \\
&\Delta_2(a \otimes b) = (\Delta_1(a))_{13}\,(\Delta_1(b))_{24} \quad \text{for all } \ a, b \in \mathrm{L}H_1' \ .
\end{aligned} \tag{7.11}
$$

By construction, we have $\Delta \circ \pi_{ij} = (\pi_{ij} \otimes \pi_{ij}) \circ \Delta_2$. So we can view π_{ij} as an injective group homomorphism $\pi_{ij} : H_2' \to \Sigma$. Note that we can naturally view $H_1' \times H_1'$ as a subgroup of H_2' and that under this identification $\pi_{ij}(a, b) = \pi_i(a) + \pi_j(b)$ for all $(a, b) \in H_1' \times H_1'$.

We denote by $K := \widehat{H}$ the group of characters on H and by $K_0 < K$ the closed subgroup of characters that are identically 1 on H_0. We identify $K_0 = \widehat{H/H_0}$. Whenever $\omega \in K$, we denote by $\alpha_\omega \in \mathrm{Aut}(\mathrm{L}H)$ the induced automorphism given $\alpha_\omega(u_x) = \omega(x) u_x$ for all $x \in H$. Applying α_ω in the i-th coordinate yields the automorphism $\alpha_\omega^i \in \mathrm{Aut}((\mathrm{L}H)^\Gamma)$, while applying α_ω in all coordinates yields the automorphism $\alpha_\omega^\Gamma \in \mathrm{Aut}((\mathrm{L}H)^\Gamma)$. By construction, we have that $\alpha_\omega^\Gamma \circ \pi_i = \pi_i \circ \alpha_\omega$. A given $a \in A = (\mathrm{L}H)^\Gamma$ belongs to A_0 if and only if $\alpha_\omega^\Gamma(a) = a$ for all $\omega \in K_0$.

Fix $x \in H$. Since $\Delta(A_0) \subset A_0 \mathbin{\overline{\otimes}} A_0$, the left hand side of the formulae in (7.10) is invariant under $\alpha_\omega^\Gamma \otimes \mathrm{id}$ for all $\omega \in K_0$. Since $\mathcal{U}_x$ is uniquely determined up to a scalar, it follows that $(\alpha_\omega \otimes \mathrm{id})(\mathcal{U}_x)$ is a multiple of $\mathcal{U}_x$ for every $\omega \in K_0$. So we find an element $\gamma(x) \in H/H_0$ such that

$$
(\alpha_\omega \otimes \mathrm{id})(\mathcal{U}_x) = \omega(\gamma(x))\,\mathcal{U}_x \quad \text{for all } \ \omega \in K_0 \ .
$$

When $x \in H_0$, we have that $\mathcal{U}_x \in \mathrm{L}H_0 \mathbin{\overline{\otimes}} \mathrm{L}H_0$ and hence $\gamma(x) = 0$. It follows that γ is a well-defined group homomorphism from H/H_0 to H/H_0.

The formulae in (7.10) entirely determine Δ, so it follows that

$$
(\alpha_\omega^i \otimes \mathrm{id}) \circ \Delta = \Delta \circ \alpha_{\omega \circ \gamma}^i ,
$$

for all $i \in \Gamma$ and all $\omega \in K_0$. Using Lemma 6.4, we conclude that $\gamma = \mathrm{id}$ and that every automorphism α^i_ω is induced by a character of Σ. It follows that there are group homomorphisms $\psi_i : \Sigma \to H/H_0$ such that

$$\alpha^i_\omega(v_s) = \omega(\psi_i(s))\, v_s \quad \text{for all } \; s \in \Sigma, i \in \Gamma, \omega \in K_0 \; .$$

A similar reasoning, using (7.11) instead of (7.10), provides a homomorphism $\psi : H'_2 \to H/H_0$ such that $(\alpha_\omega \otimes \mathrm{id})(v_s) = \omega(\psi(s))\, v_s$ for all $s \in H'_2$, $\omega \in K_0$.

Since $\alpha^i_\omega \circ \pi_{ij} = \pi_{ij} \circ (\alpha_\omega \otimes \mathrm{id})$, we have that $\psi_i \circ \pi_{ij} = \psi$. Since moreover $(\alpha_{\overline{\omega}} \otimes \mathrm{id})(x) = (\mathrm{id} \otimes \alpha_\omega)(x)$ for all $x \in \mathrm{L}H'_2$, we have $\alpha^j_\omega \circ \pi_{ij} = \pi_{ij} \circ (\alpha_{\overline{\omega}} \otimes \mathrm{id})$. Hence $\psi_j \circ \pi_{ij} = -\psi$. We further have that $\psi_k \circ \pi_{ij} = 0$ if $k \notin \{i,j\}$.

We already observed above that the subgroups $\pi_i(H'_1) < \Sigma$, $i \in \Gamma$, are in a direct sum position. Denote by $\Sigma_1 < \Sigma$ the subgroup generated by the $\pi_i(H'_1)$, $i \in \Gamma$. Since $\mathrm{L}H'_1 = \mathrm{L}H_0$, we have that $\mathrm{L}\Sigma_1 = (\mathrm{L}H_0)^\Gamma$. It follows that

$$\mathrm{L}\Sigma_1 = \{x \in A_0 \mid \alpha^i_\omega(x) = x \; \text{ for all } \; i \in \Gamma, \omega \in K_0\},$$

and hence $\Sigma_1 = \bigcap_{i \in I} \operatorname{Ker} \psi_i$.

Every permutation $\beta \in \operatorname{Perm}\Gamma$ defines an automorphism γ_β of $(\mathrm{L}H)^\Gamma$ by permuting the tensor factors. It follows from (7.10) that $(\gamma_\beta \otimes \gamma_\beta) \circ \Delta = \Delta \circ \gamma_\beta$, so that γ_β induces a group automorphism of Σ. By construction, we have $\gamma_\beta \circ \pi_i = \pi_{\beta(i)}$ and $\gamma_\beta \circ \pi_{ij} = \pi_{\beta(i),\beta(j)}$.

It is now easy to check that all assumptions of Lemma 7.15 are satisfied. We conclude from Lemma 7.15 that there exists an abelian group H' with subgroup $H'_0 < H'$ and a G-equivariant group isomorphism $p_{H'}^{-1}(H'_0) \to \Sigma$.

Proving that the isomorphism π is of the required form

We put $\mathcal{H}' := H'^{(\Gamma)}$ and $\mathcal{H}'_0 := p_{H'}^{-1}(H'_0)$. Precomposing the original identification of $\mathrm{L}\Sigma$ and $\mathrm{L}\mathcal{H}_0$, with the above identification of $\mathrm{L}\Sigma$ and $\mathrm{L}\mathcal{H}'_0$, we have brought us to the point where $\Lambda = \mathcal{H}'_0 \rtimes G$ and where the isomorphism

$$\pi : \mathrm{L}(\mathcal{H}'_0 \rtimes G) \to \mathrm{L}(\mathcal{H}_0 \rtimes G)$$

satisfies $\pi(\mathrm{L}\mathcal{H}'_0) = \mathrm{L}\mathcal{H}_0$ and $\pi(u_g) = u_g$ for all $g \in G$.

Denote by $\varphi : \mathrm{L}\mathcal{H}'_0 \to \mathrm{L}\mathcal{H}_0$ the restriction of π to $\mathrm{L}\mathcal{H}'_0$. Note that φ is a G-equivariant $*$-isomorphism. To conclude the proof of Theorem 7.1, it remains to prove that φ must be of the following special form: there exist a group isomorphism $\gamma : H'/H'_0 \to H/H_0$, a G-invariant character $\mu : \mathcal{H}_0 \to \mathbb{T}$ and a trace-preserving $*$-isomorphism $\varphi_0 : \mathrm{L}H' \to \mathrm{L}H$ such that $\varphi_0 \circ \alpha_{\omega \circ \gamma} =$

$\alpha_\omega \circ \varphi_0$ for all $\omega \in \widehat{H/H_0}$ and such that $\varphi = \alpha_\mu \circ \varphi_0^\Gamma$. Here the $*$-isomorphism $\varphi_0^\Gamma : (\mathrm{L}H')^\Gamma \to (\mathrm{L}H)^\Gamma$ is defined as the infinite tensor product of copies of φ_0.

Denote $K = \widehat{H}$, $K' = \widehat{H'}$, $K_0 = \widehat{H/H_0}$ and $K_0' = \widehat{H'/H_0'}$. Consider the compact group K^Γ and embed K_0 as a subgroup of K^Γ diagonally. We similarly consider $K_0' < (K')^\Gamma$. We identify

$$\mathrm{L}\mathcal{H}_0' = \mathrm{L}^\infty\Big(\frac{(K')^\Gamma}{K_0'}\Big) \quad \text{and} \quad \mathrm{L}\mathcal{H}_0 = \mathrm{L}^\infty\Big(\frac{K^\Gamma}{K_0}\Big) .$$

We can then view $\varphi = \theta_*$ where θ is a probability measure preserving, G-equivariant isomorphism

$$\theta : \frac{(K')^\Gamma}{K_0'} \to \frac{K^\Gamma}{K_0} .$$

Consider the natural actions $G \times K_0' \curvearrowright (K')^\Gamma$ and $G \times K_0 \curvearrowright K^\Gamma$. By Popa's cocycle superrigidity theorem [Po06b, Theorem 1.1] and [PV06, Lemma 5.2], there exist a p.m.p. isomorphism $\widetilde{\theta} : (K')^\Gamma \to K^\Gamma$, a group homomorphism $\beta : G \to K_0 : g \mapsto \beta_g$ and a continuous group isomorphism $\widehat{\gamma} : K_0' \to K_0$ such that

$$\widetilde{\theta}((g,k)\cdot\omega) = (g, \beta_g \widehat{\gamma}(k)) \cdot \widetilde{\theta}(\omega) \quad \text{and} \quad \widetilde{\theta}(\omega) + K_0 = \theta(\omega + K_0') , \qquad (7.12)$$

for all $(g,k) \in G \times K_0'$ and a.e. $\omega \in (K')^\Gamma$.

Fix $x \in H$ and denote $F_x : K^\Gamma \to \mathbb{T} : F_x(\omega) = \omega_e(x)$. As before, denote by $\delta : \Gamma \to G : \delta(g) = (g,g)$ the diagonal embedding. One checks that

$$(F_x \circ \widetilde{\theta})(\delta(g)\cdot\omega) = \beta_g(x)\,(F_x \circ \widetilde{\theta})(\omega) \;\text{ for all }\; g \in \Gamma \;\text{ and a.e. }\; \omega \in (K')^\Gamma .$$

Since Γ is i.c.c., it follows from Lemma 2.20 that the action of $\delta(\Gamma)$ on $(K')^{\Gamma - \{e\}}$ is weakly mixing, so that the function $\omega \mapsto (F_x \circ \widetilde{\theta})(\omega)$ only depends on the coordinate ω_e. Since this holds for all $x \in H$, we find a p.m.p. isomorphism $\theta_0 : K' \to K$ such that $(\widetilde{\theta}(\omega))_e = \theta_0(\omega_e)$ for a.e. ω. By construction, we have $\theta_0(k + \omega) = \widehat{\gamma}(k) + \theta_0(\omega)$ for all $k \in K_0'$ and a.e. $\omega \in K'$. Writing $\varphi_0 := (\theta_0)_*$, we obtain the trace-preserving $*$-isomorphism $\varphi_0 : \mathrm{L}H' \to \mathrm{L}H$ satisfying $\varphi_0 \circ \alpha_{\omega\circ\gamma} = \alpha_\omega \circ \varphi_0$ for all $\omega \in \widehat{H/H_0}$.

Evaluating (7.12) in the coordinate e, we find that $\beta_{\delta(g)} = 0$ for all $g \in \Gamma$, so that $\beta_{(g,h)} = \rho_g - \rho_h$ for a group homomorphism $\rho : \Gamma \to K_0 : g \mapsto \rho_g$. We also find that $\widetilde{\theta}(\omega)_g = \theta_0(\omega_g) + \rho_g$ for all $g \in \Gamma$ and a.e. $\omega \in (K')^\Gamma$. Define $\mu \in K^\Gamma / K_0$ as $\mu := (\rho_g)_{g\in\Gamma} + K_0$. Then μ is a G-invariant element of K^Γ/K_0, i.e. a G-invariant character on $\mathcal{H}_0$. By construction, we have that $\varphi = \alpha_\mu \circ \varphi_0^\Gamma$. $\square$

A combinatorial lemma

Whenever I is a countable set and H is a countable abelian group with subgroup $H_0 < H$, we consider the direct sum $H^{(I)}$, the group homomorphism

$$p_H : H^{(I)} \to H : p_H(x) = \sum_{g \in I} x_g$$

and the subgroup $p_H^{-1}(H_0)$ of $H^{(I)}$. The group $\operatorname{Perm} I$ of all permutations of I acts on $H^{(I)}$ by group automorphisms that leave the subgroup $p_H^{-1}(H_0)$ globally invariant.

For every $i \in I$, we have a natural embedding $\mu_i : H_0 \to p_H^{-1}(H_0)$ of H_0 into the i-th coordinate. Writing $(H \times H)_{H_0} := \{(x, y) \in H \times H \mid x + y \in H_0\}$, we also have natural embeddings $\mu_{ij} : (H \times H)_{H_0} \to p_H^{-1}(H_0)$ into the i-th and j-th coordinate, whenever i and j are distinct elements of I. The subgroups $\mu_{ij}((H \times H)_{H_0})$ generate $p_H^{-1}(H_0)$.

The following elementary lemma abstractly characterizes this whole setup. The lemma is actually much more awkward to state than to prove.

Lemma 7.15. *Let Σ be a countable abelian group and I a countably infinite set. Assume that we are given the following data:*

- *countable abelian groups H_1 and H_2 such that $H_1 \times H_1 < H_2$,*
- *for all $i \in I$, an injective homomorphism $\pi_i : H_1 \to \Sigma$,*
- *for all distinct $i, j \in I$, an injective homomorphism $\pi_{ij} : H_2 \to \Sigma$,*
- *an abelian group L and, for all $i \in I$, a group homomorphism $\psi_i : \Sigma \to L$,*
- *a group homomorphism $\psi : H_2 \to L$,*
- *an action of the group of all permutations $\beta \in \operatorname{Perm} I$ by group automorphisms γ_β of Σ,*

such that the following conditions hold:

- *the subgroups $\pi_{ij}(H_2)$ generate Σ,*
- *the subgroups $\pi_i(H_1)$ are in a direct sum position inside Σ and generate a subgroup of Σ denoted by Σ_1,*
- *$\pi_{ij}(a, b) = \pi_i(a) + \pi_j(b)$ for all $(a, b) \in H_1 \times H_1 \subset H_2$,*

- $\psi_i \circ \pi_{ij} = \psi = -\psi_j \circ \pi_{ij}$,
- $\psi_k \circ \pi_{ij} = 0$ *if* $k \notin \{i,j\}$,
- $\Sigma_1 = \bigcap_{i \in I} \operatorname{Ker} \psi_i$,
- *for every* $\beta \in \operatorname{Perm} I$, *we have* $\gamma_\beta \circ \pi_i = \pi_{\beta(i)}$ *and* $\gamma_\beta \circ \pi_{ij} = \pi_{\beta(i),\beta(j)}$.

Then there exist a countable abelian group H, *with a subgroup* $H_0 < H$, *and group isomorphisms*

$$\delta_1 : H_0 \to H_1 \quad , \quad \delta_2 : (H \times H)_{H_0} \to H_2 \quad \textit{and} \quad \delta : p_H^{-1}(H_0) \to \Sigma$$

such that, using the notations μ_i *and* μ_{ij} *introduced before the lemma, we have*

- δ *conjugates the actions of* $\operatorname{Perm} I$,
- $\delta \circ \mu_i = \pi_i \circ \delta_1$,
- $\delta \circ \mu_{ij} = \pi_{ij} \circ \delta_2$.

Proof. We may assume that $I = \mathbb{N}$. Since the subgroups $\pi_i(H_1) < \Sigma$ are in a direct sum position, we can assemble the π_i into an isomorphism $\pi : H_1^{(\mathbb{N})} \to \Sigma_1$. Note that π conjugates the natural actions of $\operatorname{Perm} \mathbb{N}$.

Fix $x \in H_2$. Observe that $y := \pi_{12}(x) + \pi_{23}(x) + \pi_{31}(x)$ belongs to the kernel of all ψ_i, $i \in \mathbb{N}$. Hence, $y = \pi(z)$ for some element $z \in H_1^{(\mathbb{N})}$. It follows that z is invariant under cyclic permutations of $(1,2,3)$. It also follows that z is invariant under all permutations that fix 1, 2 and 3. Since there are only finitely many $k \in \mathbb{N}$ with $z_k \neq 0$, we conclude that y must be of the form $y = \pi_1(\rho(x)) + \pi_2(\rho(x)) + \pi_3(\rho(x))$, where $\rho : H_2 \to H_1$ is a group homomorphism. Also note that $\rho(a,b) = a + b$ for all $(a,b) \in H_1 \times H_1 \subset H_2$.

We define $H := \operatorname{Ker} \rho$. We define the subgroup $H_0 < H$ given by $H_0 := \{(a,-a) \mid a \in H_1\}$. We denote $\delta_1 : H_0 \to H_1 : \delta_1(a,-a) := a$.

By construction, we have that $\pi_{12}(x) + \pi_{23}(x) + \pi_{31}(x) = 0$ for all $x \in H$. Applying γ_β for an arbitrary permutation β of $\mathbb{N}$, it follows that

$$\pi_{ij}(x) + \pi_{jk}(x) + \pi_{ki}(x) = 0 \tag{7.13}$$

for all $x \in H$ and all distinct $i,j,k \in \mathbb{N}$.

Fix $x \in H_2$. Observe that $y := \pi_{12}(x) + \pi_{21}(x)$ belongs to the kernel of all ψ_i, $i \in \mathbb{N}$. We also have that $\gamma_\beta(y) = y$ when β is the permutation of $\mathbb{N}$ that flips 1 and 2, as well as when β is a permutation that fixes 1 and 2. Reasoning as above, it follows that $\pi_{12}(x) + \pi_{21}(x) = -\pi_1(\eta(x)) - \pi_2(\eta(x))$, where $\eta : H_2 \to H_1$ is a group homomorphism. We only introduced the minus sign to make the following computation easier. Applying γ_β for an arbitrary permutation β of $\mathbb{N}$, we get that

$$\pi_{ji}(x) = -\pi_{ij}(x) + \pi_i(\eta(x)) + \pi_j(\eta(x))$$

for all $x \in H_2$ and all distinct $i, j \in \mathbb{N}$.

We prove that $\eta(x) = 0$ for all $x \in H$. Fix $x \in H$ and consider the element

$$y := \pi_{12}(x) + \pi_{23}(x) + \pi_{34}(x) + \pi_{41}(x) \; .$$

A first computation, using (7.13), yields

$$\begin{aligned} y &= -\pi_{31}(x) + \pi_{34}(x) + \pi_{41}(x) \\ &= \pi_1(\eta(x)) + \pi_3(\eta(x)) + \pi_{13}(x) + \pi_{34}(x) + \pi_{41}(x) \\ &= \pi_1(\eta(x)) + \pi_3(\eta(x)) \; . \end{aligned}$$

An analogous second computation gives

$$\begin{aligned} y &= \pi_{12}(x) - \pi_{42}(x) + \pi_{41}(x) \\ &= \pi_2(\eta(x)) + \pi_4(\eta(x)) + \pi_{12}(x) + \pi_{24}(x) + \pi_{41}(x) \\ &= \pi_2(\eta(x)) + \pi_4(\eta(x)) \; . \end{aligned}$$

Since the groups $\pi_i(H_1)$ are in a direct sum position inside Σ, both computations together imply that $\eta(x) = 0$ for all $x \in H$. It follows that $\pi_{ij}(x) = -\pi_{ji}(x)$ for all $x \in H$ and all distinct $i, j \in \mathbb{N}$. In combination with (7.13), we get that

$$\pi_{ij}(x) + \pi_{jk}(x) = \pi_{ik}(x) \tag{7.14}$$

for all $x \in H$ and all distinct $i, j, k \in \mathbb{N}$.

We claim that the homomorphism

$$\delta_2 : (H \times H)_{H_0} \to H_2 : \delta_2(x, y) = x + (0, \delta_1(x + y))$$

is an isomorphism of groups satisfying $\delta_2(x, y) = (\delta_1(x), \delta_1(y))$ for all $(x, y) \in H_0 \times H_0$. This last formula is immediate. It already implies that the image of δ_2 contains both H and $H_1 \times H_1$. Since for every $x \in H_2$, we have that $x - (0, \rho(x)) \in H$, the surjectivity of δ_2 follows. Since $\rho(\delta_2(x, y)) = \delta_1(x+y)$, the injectivity of δ_2 follows as well.

Using (7.14), it follows that the formula

$$\delta : p_H^{-1}(H_0) \to \Sigma : \delta(x) = \pi_{n+1}(\delta_1(p_H(x))) + \sum_{i=1}^{n} \pi_{i,n+1}(x_i),$$

whenever $x_k = 0$ for all $k > n$, is independent of the choice of n and hence a well-defined homomorphism satisfying $\delta \circ \mu_{ij} = \pi_{ij} \circ \delta_2$ and $\delta \circ \mu_i = \pi_i \circ \delta_1$. It immediately follows that δ conjugates the respective actions of $\operatorname{Perm} \mathbb{N}$ and that δ is surjective.

To prove the injectivity of δ, we first claim that $H_0 = H \cap \operatorname{Ker} \psi$. The direct inclusion is obvious. Conversely, assume that $y \in H$ and $\psi(y) = 0$. Put $z = \pi_{12}(y)$. We get that $z \in \operatorname{Ker} \psi_k$ for all $k \in \mathbb{N}$. So $z \in \Sigma_1$. Since $\gamma_\beta(z) = z$ for every permutation β that fixes 1 and 2, we find that $y \in H_1 \times H_1$. Since $y \in H$, we obtain the claim that $y \in H_0$. If now $\delta(x) = 0$, we get that $\psi(x_i) = \psi_i(\delta(x)) = 0$ for all $i \in \mathbb{N}$. So x belongs to $H_0^{(I)}$. Since $\delta \circ \mu_i = \pi_i \circ \delta_1$, the restriction of δ to $H_0^{(I)}$ is injective. $\square$

Proof of Theorem 1.2

Proof of Theorem 1.2. A hyperbolic group Γ has only finitely many conjugacy classes of finite subgroups (see e.g. [Br99]). By Selberg's lemma [Se60], a finitely generated linear group Γ (over a field of characteristic zero) has a finite index subgroup that is torsion-free. In both cases, Γ admits a bound on the possible orders of its finite subgroups. By the work of [CH88], [Sk88], [Oz03], [Oz07] (see [PV12, Lemma 2.4] for a more detailed explanation), we also have in both cases that Γ is weakly amenable and that Γ belongs to class $\mathcal{S}$. So every group Γ that appears in Theorem 1.2 satisfies the conditions of Theorem 7.1.

The conclusion of Theorem 7.1 describes the given $*$-isomorphism $\pi : \mathrm{L}\Lambda \to (\mathrm{L}\mathcal{G}_0)^r$ as a composition of an inner automorphism, "group like" isomorphisms implemented by group isomorphisms and characters, and the $*$-isomorphism π_θ that need not be group like in general. We now prove that in the situation of Theorem 1.2, also π_θ is group like.

1. Assume that $H = \mathbb{Z}/n\mathbb{Z}$ with $n \in \{2, 3\}$ and put $\mathcal{G} = H^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$. We apply Theorem 7.1 with $H_0 = H$. This provides an abelian group H' with $|H'| = |H|$. So, $H' \cong H$ and we may assume that $H' = H$. It only remains to prove that the automorphism $\pi_\theta : \mathrm{L}\mathcal{G} \to \mathrm{L}\mathcal{G}$ is group like. But since $\mathrm{L}H$ has dimension 2 or 3, one can check that every automorphism $\theta : \mathrm{L}H \to \mathrm{L}H$ is of the form $\theta = \alpha_\omega \circ \pi_\delta$ for some character $\omega \in \widehat{H}$ and group automorphism $\delta : H \to H$. Then π_θ is group like as well.

2. We apply Theorem 7.1 with $H_0 = \{0\}$. Since $H' \cong H$, we may assume that $H' = H$. Then $\theta : \widehat{H} \to \widehat{H}$ is a p.m.p. isomorphism satisfying $\theta(k+\omega) = k+\theta(\omega)$ for a.e. $k, \omega \in \widehat{H}$. So we find a fixed $\omega_0 \in \widehat{H}$ such that $\theta(\omega) = \omega + \omega_0$ for a.e.

$\omega \in \widehat{H}$. But then π_θ is the identity map. $\square$

Proof of Remark 1.3. Assume that Γ has no non-trivial characters. Put $G = \Gamma \times \Gamma$, $\mathcal{H}_0 = p_H^{-1}(\{0\})$ and $\mathcal{G}_0 = \mathcal{H}_0 \rtimes G$. Put $K = \widehat{H}$. Since G has no non-trivial characters, we only need to prove that $\mathcal{H}_0$ has no non-trivial G-invariant characters. This means that we have to prove that the action of G on the compact space K^Γ/K only has 0 as a fixed point. One checks that the G-fixed points in K^Γ/K are precisely the points $(\alpha_g)_{g\in\Gamma} + K$ where $\alpha : \Gamma \to K$ is a homomorphism. Since Γ has no non-trivial characters and K is abelian, such a homomorphism is constantly equal to 0. $\square$

Conclusion and future perspectives

One of the main problems in the theory of von Neumann algebras is to classify the group II_1 factors $\mathrm{L}\Gamma$ in terms of the group Γ. More precisely, we are interested in answering the following question: does the group factor $\mathrm{L}\Gamma$ *remember* the group Γ? This natural question leads to two important concepts: *softness*, this is when $\mathrm{L}\Gamma$ does not remember the group Γ, and *rigidity*, when $\mathrm{L}\Gamma$ completely remembers the group Γ. In the first case, there is a long list of examples of groups that are soft, containing all i.c.c. amenable groups [Co76]. On the other hand, it is a famous open problem whether the free group factors $\mathrm{L}\mathbb{F}_n$, with $n \geq 2$, are isomorphic or not. Another important open problem is *Connes' rigidity conjecture* [Co80a], [Co80b]: any two i.c.c. property (T) groups Γ and Λ, with isomorphic group von Neumann algebras $\mathrm{L}\Gamma \cong \mathrm{L}\Lambda$, must be isomorphic. This conjecture remains wide open, even for classical groups like $\mathrm{SL}(n, \mathbb{Z})$, with $n \geq 3$.

Ioana, Popa and Vaes [IPV10] established the first *W*-superrigidity* theorem for group von Neumann algebras: for a large class of generalized wreath product groups $G = (\mathbb{Z}/2\mathbb{Z})^{(I)} \rtimes \Gamma$, it was shown that if $\mathrm{L}G \cong \mathrm{L}\Lambda$, for an arbitrary countable group Λ, then G must be isomorphic with Λ. Such a group G is said to be *W*-superrigid* and in this case the group von Neumann algebra $\mathrm{L}G$ completely remembers G.

Motivated by the work of Ioana, Popa and Vaes, we have found in this thesis more natural examples of W*-superrigid groups. Given a countable group Γ, we consider the action of the direct product $\Gamma \times \Gamma$ on Γ by left-right multiplication and we define the generalized wreath product group $\mathcal{G} := H^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$, where $H = \mathbb{Z}/2\mathbb{Z}$. We prove that $\mathcal{G}$ is W*-superrigid whenever Γ belongs to a large class of non-amenable groups, containing free groups, hyperbolic groups, non-trivial free products, certain groups with positive first ℓ^2-Betti number, etc.

As we have already remarked in Chapter 1, not all non-amenable left-right wreath product $\mathcal{G} = H^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$ are W*-superrigid, but we think one can prove that $\mathcal{G}$ is W*-superrigid for many other non-amenable groups than those covered in this thesis (e.g. groups with property (T), groups with positive first ℓ^2-Betti number, etc.)

On the other hand, we hope to prove W*-superrigidity also for semi-direct products coming from a different type of actions $\Gamma \curvearrowright H$. For instance, we can consider Bernoulli actions $\Gamma \curvearrowright H^{(I)}$ of groups with certain rigidity property (e.g. Γ is a free product of property (T) groups). Under suitable assumptions for the action $\Gamma \curvearrowright I$, one might get that the wreath product $\mathcal{G} = H^{(I)} \rtimes \Gamma$ is W*-superrigid. More generally, one can investigate actions $\Gamma \curvearrowright H$ that are not of Bernoulli type, but which have similar properties. The main difficulty here is to generalize the conjugacy criterion for group actions [IPV10, Theorem 6.1] to this kind of actions.

Another class of actions that we would like to investigate further consists of *profinite actions* with spectral gap. Popa [Po09] proved rigidity results for II$_1$ factors that are inductive limits of subfactors with spectral gap. Ioana [Io13] studied actions with spectral gap of residually finite groups on their profinite completions and proved that under certain circumstances they are orbit equivalence rigid. There is a strong connection between profinite actions with spectral gap and a weaker form of property (T), called *property* (τ). Using the fact that $\mathbb{F}_n$, $n \geq 2$, and $\mathrm{SL}_2(\mathbb{Z})$ enjoy this weaker form of property (T), Ioana provided explicit uncountable families of free ergodic p.m.p. actions of $\mathbb{F}_n$ and $\mathrm{SL}_2(\mathbb{Z})$ that are pairwise non orbit equivalent. In particular, by the work of Ozawa and Popa, all these actions give non-isomorphic II$_1$ factors. We hope that we can speculate the rigidity properties that this class of actions manifests in order to prove certain W*-superrigidity results.

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